The Hierarchy of Topological Dynamical Systems with Discrete Spectrum

– REVISED VERSION OF THE MASTER THESIS –

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Abstract

Equicontinuous dynamical systems are *“non-chaotic”* in many ways. For example their states continuously depend on initial conditions and have zero entropy. In the case of Abelian acting groups each point exhibits the same dynamical behavior. This means that any two points can be exchanged by symmetries, i.e. automorphisms of the equicontinuous system. There is a complete classification of Abelian equicontinuous systems as group compactifications. In the non-Abelian case the situation requires more sophisticated tools but also justifies equicontinuous systems to be interpreted as *“simple”*.

It is well-known that any (compact) topological dynamical system has an equicontinuous factor maximal among all equicontinuous factors. The invertibility properties as well as other regularity properties of the factor map onto this maximal equicontinuous factor (MEF) can tell us how far a system is away from the simple equicontinuous case. In fact, the literature offers a variety of equivalences between dynamical properties on the one hand and the invertibility properties of the factor map to the MEF on the other hand. Those invertibility properties can be put into a canonical hierarchy starting from homeomorphy. Via those aforementioned equivalences, a parallel hierarchy of dynamical properties starting with equicontinuity emerges. This hierarchy can also be understood as a map of the many different topological models for a measure preserving dynamical system with discrete spectrum.

Our aim here is to present a collective overview of the current state of knowledge.

Zusammenfassung

Ein gleichgradig stetiges System erfüllt sehr viele Interpretationen von *“nicht chaotisch”*. Beispielsweise hängen die Zustände eines solchen Systems stetig von den Anfangsbedingungen ab und es hat Entropie Null. Falls die operierende Gruppe Abelsch ist, so kann man sogar sagen, dass alle Punkte das selbe dynamische Verhalten zeigen. Das bedeutet, dass es für je zwei Punkte eine Symmetrie, d.h. einen Automorphismus des Systems, gibt, die diese tauscht. Weiterhin gibt es eine vollständige Klassifizierung von solchen Abelschen gleichgradig stetigen Systemen als Kompaktifizierungen von Gruppen. In dem Fall einer nicht-Abelschen Gruppe wird die Maschinerie aufwändiger. Die Interpretation, dass gleichgradig stetige Systeme *“einfach”* sind, bleibt aber gerechtfertigt.

Ein bekanntes Resultat besagt, dass jedes (kompakte) topologische dynamische System einen gleichgradig stetigen Faktor besitzt der maximal unter allen gleichgradig stetigen Faktoren ist. Die Invertierbarkeits- und sonstigen Regularitätseigenschaften der Faktorabbildung auf diesen maximalen gleichgradig stetigen Faktor (MEF) sagt uns wie weit ein System von gleichgradiger Stetigkeit entfernt ist. Diese Heuristik wird von der Literatur mannigfaltig bestätigt. Viele Äquivalenzen zwischen dynamischen Eigenschaften auf der einen Seite und Invertierbarkeitseigenschaften auf der anderen Seite sind bekannt. Invertierbarkeitseigenschaften lassen sich beginnend bei Homöomorphie in eine kanonische Hierarchie bringen. Die besagten Äquivalenzen lassen nun eine parallele Hierarchie von dynamischen Systemen ausgehend von gleichgradiger Stetigkeit entstehen.

Mit dieser Arbeit soll ein Überblick über den kollektiven aktuellen Stand des Wissens gegeben werden.

Contents

0 Introduction

Let G be a topological group acting continuously on a topological space X by the action

$$\alpha : G \times X \longrightarrow X, (g, x) \longmapsto \alpha(g, x) \,.$$

Dynamical systems like the triple (X, G, α) are studied extensively in the eponymous field of mathematics.

The notion of *"chaos"* is an important concept in this context. Much work has been carried out to characterize the *"chaoticity"* of dynamical systems. Robert L. Devaney defines *"Chaos"* in [Devaney, 1990] with three properties:

i) sensitivity to initial conditions,

ii) topological transitivity,

iii) and the set of periodic points is dense.

Equicontinuous Systems[1] are systems with *continuous* dependence on initial conditions. So (besides trivial cases) they fail condition *(i)*. The condition *(ii)* is more of technical nature as by restricting the phase space X to the orbit closure every point can be viewed as part of a transitive system. We will see[2] that, at least in the case of G being Abelian and X being metric, for transitive equicontinuous systems condition *(iii)* is equivalent to the system itself being periodic.

Hence, equicontinuous systems strongly fail the requirements for chaos. In fact they are the prime example of *"ordered"* systems. A paradigmatic ordered system is the rotation of the circle by a fixed degree. This circle rotation is part of the whole class of rotations on group compactifications.[3] We will learn that every equicontinuous system with Abelian acting group G is conjugate to a rotation on a group compactification.[4]

An idea on how to study *"un-ordered"* systems could be to somehow measure how far a system is away from being equicontinuous. However, at first sight it is not very clear what one could mean by this. Yet, it turns out that every system has a *maximal equicontinuous factor*.[5] One can understand a factor as a simplification of a system by looking at it with a coarser resolution: combining points into equivalence classes. Obviously one can always find equicontinuous factors by brute force, for example by combining *all* points into *one* equivalence class. The *maximality* of the *maximal equicontinuous factor* means that one only simplifies just as much as it is necessary in order to obtain an equicontinuous factor. Now one can try to learn how *complex* or *chaotic* a system is by looking at how much simplification is needed in order to obtain an equicontinuous factor. Since this process is encoded in the factor map, we study the regularity properties of the latter.

The idea that a factor corresponds to some sort of coarser resolution on the image leads to the question *"How big are the fibers of the factor map?"* or *"How do we*

[1] See Definition 1.11.6.

[2] See Theorem 4.2.2.

[3] See Definition 1.11.16.

[4] See Theorem 4.4.1.

[5] See Theorem 2.2.1.

have to restrict the domain and co-domain of a factor map in order for it to become invertible". We study four such invertibility properties and will identify corresponding dynamical properties. The invertibility properties have a clear hierarchy and we thus obtain a parallel hierarchy of dynamical properties.

There is a second viewpoint on that hierarchy: All the invertibility properties studied here imply that the systems are measure theoretically isomorphic to a measure preserving system with discrete spectrum. This means that they are all *topological models* of systems with discrete spectrum. Fixing one such discrete spectrum measure preserving dynamical system our hierarchy turns into a map of all the different topological properties that topological models can have.

Part I
Background Knowledge

1 Preliminaries

Here I will present general background knowledge for the convenience of the reader. Most of the results stated here are well-known however often we need a special formulation for our purposes. If I was able to find the result in standard literature I provide reference. Sometimes if the proof has a simple and enlightening idea at it's core a sketch will be provided. In some cases no reference that suits our purpose could be found, then I provide the proof by myself.

1.1 Notation

Given a set X we denote by $\mathfrak{P}(X)$ its power set. If Y is another set we denote the set of functions from X to Y by $\mathcal{F}(X,Y)$ or Y^X. Given a function f we denote by $\operatorname{dom}(f)$ its domain, i.e. the set of arguments for which f is defined.

Let P be a predicate. We will sometimes write *"x : $\mathsf{P}(\mathsf{x})$"* as a shorthand for *"x such that $\mathsf{P}(\mathsf{x})$"*. Given any algebraic structure A we write $B \leq A$ whenever B is a substructure of A. We will use the same notation for sub-σ-algebras. Let $\operatorname{proj}_A : A \times B \to A$ denote the projection onto A. Given a subset $R \subseteq X \times Y$ we define

$$R[x] := \{y \in Y \mid (x,y) \in R\} .$$

1.2 Category Theory

This work makes use of the language of category theory to speak more easily about about certain ideas while remaining precise.

For the convenience of readers unfamiliar with category theoretic notions, we include some basic definitions here.

1.2.1 Basic Concepts

Definition 1.2.1 (Category)**.** A **category** C consists of a class of objects $\mathrm{ob}(\mathsf{C})$ together with classes of morphisms $\mathrm{Hom}(A, B)$ for each $A, B \in \mathrm{ob}(\mathsf{C})$ and an operation

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \longrightarrow \mathrm{Hom}(A, C),\ (f, g) \longmapsto g \circ f$$

for each $A, B, C \in \mathrm{ob}(\mathsf{C})$, such that

Associativity: For each 4-tuple of objects $A, B, C, D \in \mathrm{ob}(\mathsf{C})$ and each chain of morphisms $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$ connecting the four we have

$$(h \circ g) \circ f = h \circ (g \circ f)\,.$$

Identity For each $A \in \mathrm{ob}(\mathsf{C})$ there is $f \in \mathrm{Hom}(A, A)$ such that for any $B \in \mathrm{ob}(\mathsf{C})$ we have $g \circ f = g$ for $g \in \mathrm{Hom}(A, B)$ as well as $f \circ h = h$ for $h \in \mathrm{Hom}(B, A)$. It turns out that such an f is unique and we will write $f =: \mathrm{Id}_A$.

Usually we will omit the ob and write $A \in \mathsf{C}$. Further we often write $f : A \to B$ instead of $f \in \mathrm{Hom}(A, B)$.

Given a category C, we denote by $\mathrm{Hom}_{\mathsf{C}}(A, B)$ the class of morphisms from A to B.

Definition 1.2.2 (Subcategory)**.** Let C be a category. Let S be a category such that $\mathrm{ob}(\mathsf{S}) \subseteq \mathrm{ob}(\mathsf{C})$ and

$$\mathrm{Hom}_{\mathsf{S}}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$$

for any $A, B \in \mathrm{ob}(\mathsf{S})$. We call S a **subcategory** and write $\mathsf{S} \subseteq \mathsf{C}$.

Definition 1.2.3 (Full Subcategory)**.** A subcategory $\mathsf{S} \subseteq \mathsf{C}$ is called **full** if and only if

$$\mathrm{Hom}_{\mathsf{S}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$$

for any $A, B \in \mathrm{ob}(\mathsf{S})$.

Remark 1.2.3.1. A full subcategory only removes objects and not morphisms.

The categories we consider here are all of a special form: The objects can be understood as sets with additional structure and the morphisms are functions between these sets preserving the additional structure. The composition $\circ$ of morphisms will always be the regular composition of functions.

Let us consider some examples of such categories:

Example 1.2.3.1. a) The category Set has all sets as objects. A morphism from a set A to a set B is just a function $f : A \to B$.

b) The category Top contains all topological spaces as objects and the continuous functions as morphisms.

c) The full subcategory $\mathsf{CHaus} \subset \mathsf{Top}$ consists of all compact Hausdorff spaces.

d) The category Meas consists of all the measure spaces as objects and has measurable measure-preserving maps as the morphisms.

e) The full subcategory $\mathsf{Prob} \subset \mathsf{Meas}$ consists of all probability spaces.

f) The main objects for topological dynamics, smooth dynamics and ergodic theory etc. can be unified category theoretically in the following way: Fix a group G and a category C. We define a category $\mathsf{CDyn}(G)$ where objects are pairs (A, α) where $A \in \mathsf{C}$ and $\alpha : G \to \mathrm{Hom}_{\mathsf{C}}(A, A)$ is a group homomorphism.[6] A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is a morphism between (A, α) and (B, β) if and only if

$$\forall g \in G : f \circ \alpha(g) = \beta(g) \circ f \, .$$

We call such morphisms between dynamical systems factor maps.

A useful idea from category theory which simplifies the mathematical language is the notion of a functor.

Definition 1.2.4 (Functor)**.** A **(covariant) functor** F is a map sending the objects of one category C to objects of another category D together with mappings (also called F)

$$F : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(F(A), F(B)),\ f \longmapsto F(f)$$

such that $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$. If $F(f \circ g) = F(g) \circ F(f)$ instead of $F(f \circ g) = F(f) \circ F(g)$ F is called a **contravariant functor**.

Functors thus allow to translate from one category to another. The covariance leads to many other preserved structures. However, not every category theoretical notion is preserved by functors.

There are many correspondences between objects of different categories. The relevant functoriality is that there is additionally a transformation of morphisms in a covariant (or contravariant) way.

Remark 1.2.4.1. We want to distinguish between a group G and the underlying set, between a topological space and the underlying set and so on. Within set theory this is done by pairing the underlying set with the group operation, the topology and so on. However a group homomorphism, or a continuous map etc. is set theoretically just a map between sets. In many cases it is very useful to view the structure preserving maps as a subset of the set of all maps between two sets. Category theoretically we differentiate fundamentally between objects and morphisms in different categories. For some categories morphisms between objects can be purely abstract and may have no representation as a map between sets. Category theory also offers us the tool, functors, in order to talk about how to view objects and morphisms in different categories.

In this work we only consider categories which can be understood as sets with additional structure. Such a category is implicitly enriched with a **forgetful functor** $U_{\mathsf{C}} : \mathsf{C} \to \mathsf{Set}$ which just forgets the additional structure. For example the forgetful

[6] Clearly $\mathrm{Hom}_C(A, A)$ is not always a group, as there can be morphisms without an inverse. So α is a group homomorphism into the group of all invertible morphisms $A \to A$.

functor of the category Top forgets the topology and the one in the category Grp forgets the group operation.

Now let C and D be two categories equipped with forgetful functors $U_{\mathsf{C}} : \mathsf{C} \to \mathsf{Set}$ and $U_{\mathsf{D}} : \mathsf{D} \to \mathsf{Set}$. We can use those forgetful functors to model what we mean by the *"underlying sets of $C \in \mathsf{C}$ and $D \in \mathsf{D}$ are equal"* by $U_{\mathsf{C}}(C) = U_{\mathsf{D}}(D)$. Now suppose that the underlying sets of C, C' and D, D' are equal. We can further model the meaning of *"Those morphisms are pointwise the same"* by $U_{\mathsf{C}}(f) = U_{\mathsf{D}}(g)$.

1.2.2 Adjoints and Universal Properties

We follow [Leinster, 2014].

First we consider *"parallel"* functors. Let C and D be categories and $G, F : \mathsf{C} \to \mathsf{D}$ be two functors.

Definition 1.2.5 (Natural Transformation)**.** A family of maps

$$\alpha := (\alpha(A) : F(A) \to G(A))_{A \in \mathrm{ob}(\mathsf{C})} \in \prod_{A \in \mathrm{ob}(\mathsf{C})} \mathrm{Hom}_{\mathsf{D}}(F(A), G(A))$$

is a **natural transformation** $\alpha : F \to G$ if and only if $\alpha(A') \circ F(f) = G(f) \circ \alpha(A)$ for any $A, A' \in \mathrm{ob}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{C}}(A, A')$, i.e. the diagram in Figure 1 commutes.

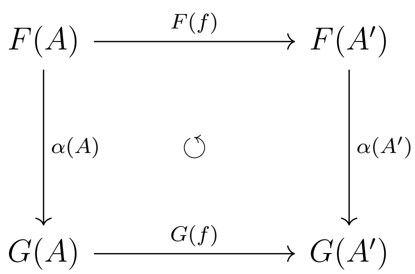

Figure 1: Natural Transformation

Definition 1.2.6 (Naturally Isomorphic)**.** Two functors $F, G : \mathsf{C} \to \mathsf{D}$ are called **naturally isomorphic** if and only if there are natural transformations $\alpha : F \to G$ and $\beta : G \to F$ such that $\alpha(A) \circ \beta(A) = \mathrm{Id}_G(A)$ and $\beta(A) \circ \alpha(A) = \mathrm{Id}_F(A)$ for all $A \in \mathrm{ob}(\mathsf{C})$.

Example 1.2.6.1. i) α is a natural isomorphism if and only if all $\alpha(A)$ are isomorphisms. This can be seen as $\beta(A) := \alpha(A)^{-1}$ yields yet again a natural transformation $\beta : G \to F$.

ii) A paradigmatic example of a natural isomorphism is the following:[7] Let k be a field and let $\mathsf{FinVec}(k)$ denote the category of finite dimensional vector spaces over k. Recall from basic linear algebra, that every finite dimensional vector

[7] See e.g. the `wikipedia` article `https://en.wikipedia.org/wiki/Natural_transformation#Double_dual_of_a_vector_space`

space $V \in \mathsf{FinVec}(k)$ is isomorphic to its bi-dual V''.[8] We can send a linear map $f : V \to W$ to a map $f' : V' \to W'$, $g \mapsto g \circ f$. So $f'' : V'' \to W''$ is given by $h \mapsto h \circ f'$.

It seems as this assignment of $V \in \mathsf{FinVec}(k)$ to the isomorphism $\iota_V : V \to V''$ is not only given vector space-wise but in a *"natural"* way for the whole category $\mathsf{FinVec}(k)$ at once. This can be made precise:

Proposition 1.2.7. *The bi-dual functor* $\mathsf{bidual} : \mathsf{FinVec}(k) \to \mathsf{FinVec}(k)$ *is naturally isomorphic to the identity functor* $\mathrm{Id}_{\mathsf{FinVec}}(k)$.

Proof. We define a natural transformation $\alpha : \mathrm{Id}_{\mathsf{FinVec}} \to \mathsf{bidual}$. Let $V \in \mathsf{FinVec}(k)$. The bi-dual V'' contains all the linear forms on the dual V'. The dual V' contains all the linear forms on the original vector space V. Clearly $e_v : V' \to k$, $f \mapsto f(v)$ is linear in f, so $e_v \in V''$. Now linear algebra tells us that $\alpha(V) : V \to V''$, $v \mapsto e_v$ is an isomorphism. It remains to show that $\alpha : \mathrm{Id} \to \mathsf{bidual}$ is a natural transformation. Let $W \in \mathsf{FinVec}(k)$ be another vector space and $f : V \to W$ linear. Then $\alpha(W) \circ f(v) = e_{f(v)}$. However for $g \in W'$ we have

$$e_{f(v)}(g) = g(f(v)) = f'(g)(v) = e_v(f' \circ g) = f''(e_v)(g) = \mathsf{bidual}(f) \circ \alpha(V)(v)\,.$$

So $\alpha(W) \circ f = \mathsf{bidual}(f) \circ \alpha(V)$. □

iii) Let C be any category and G a group. Earlier we introduced the category $\mathsf{CDyn}(G)$. Let G be a category with one object $*$ and $\mathrm{Hom}_{\mathsf{G}}(*, *) = G$. This category G clearly resembles the group G. Now consider the category of functors from G to C. This category is usually denoted by $[\mathsf{G}, \mathsf{C}]$. The morphisms between functors are the natural transformations.

$F \in [\mathsf{G}, \mathsf{C}]$ picks an object $F(*) \in \mathsf{C}$ and picks morphisms $F(g) : F(*) \to F(*)$. This yields an object $\hat{F}$ of $\mathsf{CDyn}(G)$. Clearly we can also construct an object of $[\mathsf{G}, \mathsf{C}]$ out of each object of $\mathsf{CDyn}(\mathsf{G})$. In order to see that those mappings are functorial and $[\mathsf{G}, \mathsf{C}]$ and $\mathsf{CDyn}(G)$ are equivalent it remains to show that a α between $F, H \in [\mathsf{G}, \mathsf{C}]$ (a natural transformation) corresponds to a factor map between $\hat{F}, \hat{H} \in \mathsf{CDyn}(G)$. Figure 1 turns into Figure 2. So $\pi := \alpha(*)$ is a

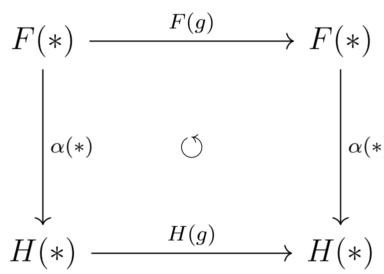

Figure 2: Natural Transformations and Factor Maps

factor map between $\hat{F}$ and $\hat{H}$.

[8]This works for any **algebraic** vector space, e.g. a vector space without norm / topology etc, however the general result is of no importance for the example.

Now we consider *"anti-parallel"* functors. Let C and D be categories and $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ be functors.

Functors can be inverse to each other. One can loosen this notion by requiring that $G(F(A)) \cong A$ for all $A \in \mathrm{ob}(\mathsf{C})$. This is however still too strict.

One very promising route is to require that the sets of morphisms $\mathrm{Hom}_{\mathsf{C}}(A, G(B))$ and $\mathrm{Hom}_{\mathsf{D}}(F(A), B)$ are in bijection to each other. We however need one additional assumption of naturality.

Definition 1.2.8 (Adjunction)**.** Suppose that

$$a : \mathrm{Hom}_{\mathsf{C}}(A, G(B)) \to \mathrm{Hom}_{\mathsf{D}}(F(A), B)$$

is a bijection for all $A \in \mathrm{ob}(\mathsf{C})$ and $B \in \mathrm{ob}(\mathsf{D})$. We call a an **adjunction** between F and G if and only if

$$a(q \circ g) = G(q) \circ a(g)$$

for all $A \in \mathrm{ob}(\mathsf{C})$, $B, B' \in \mathrm{ob}(\mathsf{D})$ and $g : F(A) \to B$, $q : B \to B'$ as well as

$$a^{-1}(f \circ p) = a^{-1}(f) \circ F(p)$$

for all $B \in \mathrm{ob}(\mathsf{D})$, $A, A' \in \mathrm{ob}(\mathsf{C})$ and $f : A \to G(B)$, $p : A' \to A$.

$$a\left(F(A) \xrightarrow{\quad g \quad} B \xrightarrow{\quad q \quad} B' \right) \qquad a^{-1}\left(A' \xrightarrow{\quad p \quad} A \xrightarrow{\quad f \quad} G(B) \right)$$

$$= \qquad\qquad\qquad\qquad =$$

$$A \xrightarrow{\ a(g)\ } G(B) \xrightarrow{G(q)} G(B') \qquad F(A') \xrightarrow{F(p)} F(A) \xrightarrow{a^{-1}(f)} B$$

Figure 3: Adjunction

If a is an adjunction between F and G we will call F a **left-adjoint** of G and G a **right-adjoint** of F.

Proposition 1.2.9. *All left-adjoints of a functor are naturally isomorphic.*[9]

Let C and D be categories.

Definition 1.2.10 (Universal Morphism)**.** Fix an object $X \in \mathsf{D}$ and a functor $F : \mathsf{C} \to \mathsf{D}$. A **universal morphism from** X **to** F is a pair (A, u), where $A \in \mathrm{ob}(\mathsf{C})$ and $u \in \mathrm{Hom}_{\mathsf{D}}(X, F(A))$, satisfying: For any $A' \in \mathrm{ob}(\mathsf{C})$ and $f \in \mathrm{Hom}_{\mathsf{D}}(X, F(A'))$ there is a *unique* morphism $h \in \mathrm{Hom}_{\mathsf{C}}(A, A')$ such that $f = F(h) \circ u$.

Construction 1.2.11. Suppose that for any $X \in \mathrm{ob}(\mathsf{D})$ the pair $(G(X), u_X)$ is a universal morphism from X to F. Let $X, Y \in \mathrm{ob}(\mathsf{D})$ and $h \in \mathrm{Hom}_{\mathsf{D}}(X, Y)$. Then $f := u_Y \circ h : X \to F(G(Y))$. By Definition 1.2.10 there is a morphism $G(h) : G(X) \to G(Y)$ such that $f = F(G(h)) \circ u_1$.

Proposition 1.2.12. *G is a left-adjoint functor to F.*[10]

[9]This is Example 4.3.13. in [Leinster, 2014, p. 105].

[10]This can be found in the literature for example by combining Theorem 5.2.1. and 5.2.2. in [Asperti and Longo, 1991].

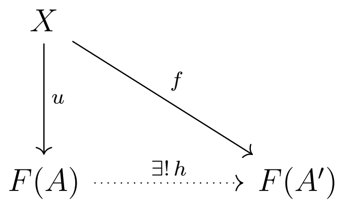

Figure 4: Universal Morphism

1.3 Ordinal Numbers

I follow closely the presentation given in [Jech, 1971]. Unlike to the rest of this work we will in this section only assume the AXIOM OF CHOICE if explicitly stated.

1.3.1 Well-Orderings

Definition 1.3.1 (Partial Order)**.** Let P be a set and $<\subseteq P \times P$. We write $a < b$ if and only if $(a, b) \in <$. We call $<$ a **partial order** if and only if

(i) For any $p \in P$ we have $p \not< p$.

(ii) For any $p, q, r \in P$ we have that whenever $p < q$ and $q < r$ then $p < r$ (Transitivity).

We then call $(P, <)$ a **partially ordered set**.

Definition 1.3.2 (Linear Order)**.** We call a partial order $<$ on P a **linear** (or sometimes **total**) order if and only if for any $p, q \in P$ we have either $p < q$ or $p = q$ or $q < p$.

Given a partially ordered set $(P, <)$ we will write $p \leqslant q$ if and only if $p = q$ or $p < q$. We also write $p > q$ to denote $q < p$ and call $>$ the reversed order. There are several notions of *"smallest"* given a partially ordered set:

Definition 1.3.3 (Least Element)**.** An element a of a partially ordered set $(P, <)$ is called a **least element** of a subset $X \subseteq P$ if and only if $a \in X$ and for any $x \in X$ we have $a \leqslant x$. A least element in reversed order is called **greatest element**.

Remark 1.3.3.1. Least elements are unique. So we can call them *the* least element.

Definition 1.3.4 (Minimal Element)**.** An element a of a partially ordered set $(P, <)$ is called a **minimal element** or **minimum** of a subset $X \subseteq P$ if and only if $a \in X$ and for any $x \in X$ we have $x \not< a$. A minimum in reversed order is called **maximal** or **maximum**.

Definition 1.3.5 (Lower Bound)**.** An element a of a partially ordered set $(P, <)$ is called a **lower bound** of a subset $X \subseteq P$ if and only if for any $x \in X$ we have $a \leqslant x$. A lower bound in reversed order is called **upper bound**.

Definition 1.3.6 (Infimum)**.** An element a of a partially ordered set $(P, <)$ is called a **infimum** of a subset $X \subseteq P$ if and only if it is a greatest element of all lower bounds of X. An infimum in reversed order is called **supremum**.

Remark 1.3.6.1. As least and greatest elements are unique, if they exists, we can speak of *the* infimum and *the* supremum.

Now in order to describe the category of partially ordered sets we need to think about morphisms.

Definition 1.3.7 (Morphisms of Partially Ordered Sets)**.** Let $(P, <)$ and $(Q, \prec)$ be partially ordered. Let $f : P \to Q$ be a function between the sets P and Q. We call f ...

... a Homomorphism if and only if for any $x, y \in P$ we have that $x \leqslant y$ implies $f(x) \preceq f(y)$.

... an Embedding if and only if f is injective and for any $x, y \in P$ we have that $x < y$ if and only if $f(x) \prec f(y)$.

... an Isomorphism if and only if f is a surjective embedding.

... Order Preserving if and only if for any $x, y \in P$ we have that $x < y$ implies $f(x) \prec f(y)$.

Now for the main type of order relevant in the theory of ordinal numbers.

Definition 1.3.8 (Well-Ordering)**.** A partially ordered set $(P, <)$ is called **well-ordered** if and only if every non-empty subset $\emptyset \neq X \subseteq P$ has a least element.

Lemma 1.3.9. *Let $(P, <)$ be well-ordered and $p \in P$. Then p is the unique least element of*

$$X_p := \{x \in P \mid x \nless p\} .$$

Proof. Let y be any least element of X_p. Then $y \in X_p$ and thus $y \nless p$. However $p \in X_p$ and as y is a least element we have $y \leqslant p$. Together we learn $y = p$. □

Lemma 1.3.10. *Any well ordered set is linearly ordered.*

Remark 1.3.10.1. In linearly ordered sets the notion of minimal element and least element coincide.

Proof. Let $(P, <)$ be well-ordered. Consider the subset of all elements which are not in relation to all other elements

$$Y := \{x \in P \mid \exists q \in P : q \nless x, x \neq q \text{ and } x \nless q\} .$$

If Y is non-empty it has an element $p \in Y$. Now consider the set X_p from the previous lemma. We know that p is the least element from X_p. Let q be an element not in relation to p. Then $q \in X_p$. However, $p \leqslant q$ as p is the least element of X_p. A contradiction. □

The idea behind ordinal numbers is that they represent isomorphism classes of well-ordered sets. As neither the isomorphism classes themselves nor the collection of them are sets an we can not employ the AXIOM OF CHOICE even if we wanted to. But even if we applied a *global* version of AC to obtain such representants we would not be satisfied, as we want to have a constructive procedure to obtain *the next larger* ordinal from a given one as well as an ordinal larger than any given set of ordinals. We thus must find a nice and natural way to constructively pick representatives of those isomorphy classes. The following definition is due to VON NEUMANN in his text [v. Neumann, 1928].

Recall that from the view-point of set theory (without URELEMENTEN) everything that sets contain are again sets. In particular it makes sense to ask whether an element of a set is also a subset.

Definition 1.3.11 (Transitive Set)**.** A set S is called **transitive** if and only if

$$\forall x \in S : x \subseteq S\,.$$

Definition 1.3.12 (VON NEUMANN Ordinal Number)**.** A set S is called an **ordinal (number)** if and only if it is a transitive set and $(S, \in)$ is well-ordered.

Lemma 1.3.13. *All ordinals contain $\emptyset$ as the least element w.r.t. $\in$.*

Proof. Let $\mathfrak{a}$ be an ordinal and $a \in \mathfrak{a}$ be the least element. Suppose that a is non-empty. Then there is x such that $x \in a$. As $\mathfrak{a}$ is transitive we have that $a \subseteq \mathfrak{a}$. Thus $x \in \mathfrak{a}$. This is a contradiction to a being the least element of $\mathfrak{a}$. □

Definition 1.3.14 (Initial Segment)**.** Let $(P, <)$ be well-ordered and $p \in P$. We define the **initial segment** of P given by p as

$$\hat{p} := \{x \in P \mid x < p\}\ .$$

Remark 1.3.14.1. Let $\mathfrak{a}$ be an ordinal and $a \in \mathfrak{a}$. Then by the order given by $\in$ we have $a = \hat{a}$.

The following is Lemma 2 in [Jech, 1971, p.7]:

Lemma 1.3.15. *If $(P, <)$ is well-ordered and $f : (P, <) \longrightarrow (P, <)$ is order preserving then for any $x \in X$ we have $x \leqslant f(x)$.*

Proof. Suppose not and let p be the least element of the set $Y := \{x \in P \mid x > f(x)\}$. Consider the initial segment $\hat{p}$. As $p \in Y$, we learn $f(p) \in \hat{p}$. As p is the least element of Y, we conclude for any $x \in \hat{p}$ that $x \leqslant f(x)$. In particular we have $f(p) \leqslant f(f(p))$. However by order-preservation of f we have $f(p) > f(f(p))$, a contradiction. □

Lemma 1.3.16. *No well-ordered set is isomorphic to an initial segment of itself.*[11]

Proof. Take any well-ordered set $(P, <)$ and $x \in P$. Suppose that $f : P \to \hat{x}$ is an isomorphism. We can view f as an order preserving map $P \to P$. Then Lemma 1.3.15 implies that $f(x) \geqslant x$ and thus $f(x) \notin \hat{x}$. A contradiction. □

[11] This is Lemma 3 in [Jech, 1971, p.7].

Lemma 1.3.17. *Let* $(P,<)$ *and* $(Q,\prec)$ *be well-ordered sets. Then they are either isomorphic or* $(P,<)$ *is isomorphic to an initial segment of* $(Q,\prec)$ *or vice versa.*[12]

Sketch of a proof. Define a relation

$$f := \{(p,q) \in P \times Q \mid \hat{p} \text{ is isomorphic to } \hat{q}\} \; .$$

Now as initial segments are well-ordered (as subsets of well-ordered sets) Lemma 1.3.16 applies and shows that for each $p \in P$ there is at most one $q \in Q$ such that $(p,q) \in f$. By the same argument f is injective. So f is an injective partial function.

We need to show that f has either full domain (i.e. $\forall p \in P : \exists q \in Q : (p,q) \in f$) or f has full range (is surjective, i.e. $\forall q \in Q : \exists p \in P : (p,q) \in f$). Assume f has neither full range nor full domain. Note that by Lemma 1.3.16 we can conclude that isomorphisms are uniquely determined. Let

$$\tilde{P} := \{p \in P \mid \hat{p} \text{ is not isomorphic to any } \hat{q} \text{ for } q \in Q\} \; .$$

Similarly define

$$\tilde{Q} := \{q \in Q \mid \hat{q} \text{ is not isomorphic to any } \hat{p} \text{ for } p \in P\} \; .$$

Let $p := \min \tilde{P}$ and $q := \min \tilde{Q}$. There is a (unique) isomorphism g between $\{p' \in P \mid p' < p\}$ and $\{q' \in Q \mid q' < q\}$. Then g can be extended by $p \mapsto q$, a contradiction.

So w.l.o.g. we can assume that f has full domain (if not consider f^{-1} and switch roles of P and Q).

Finally show that the range of f must be an initial segment. □

The previous Lemma 1.3.17 motivates the following definition

Definition 1.3.18. Given well-ordered sets $(P,<)$ and $(Q,\prec)$ we denote

$(P,<) < (Q,\prec)$ if and only if $(P,<)$ is isomorphic to an initial segment of $(Q,\prec)$.

$(P,<) \cong (Q,\prec)$ if and only if $(P,<)$ and $(Q,\prec)$ are isomorphic.

$(Q,\prec) < (P,<)$ if and only if $(Q,\prec)$ is isomorphic to an initial segment of $(P,<)$.

Lemma 1.3.19. *For two ordinals* $(\mathfrak{a},\in)$ *and* $(\mathfrak{b},\in)$ *we have* $\mathfrak{a} < \mathfrak{b}$ *if and only if* $\mathfrak{a} \in \mathfrak{b}$.

Proof. Let $\mathfrak{a} < \mathfrak{b}$. There is $b \in \mathfrak{b}$ such that there is an isomorphism

$$f : \mathfrak{a} \to \hat{b} \, .$$

We show that $\mathfrak{a} = b$. Let $W := \{p \in \mathfrak{a} \mid f(p) \neq p\}$. Assume that f is not the identity. Then W is non-empty and has a least element a. By Lemma 1.3.13 we know that $\emptyset$ is least in both $\mathfrak{a}$ and $\mathfrak{b}$. Isomorphisms map least elements to least elements. So $\emptyset \neq a$. Hence, the initial segment $\hat{a}$ is non-empty and $f|_{\hat{a}} = \mathrm{Id}_{\hat{a}}$. Clearly $\widehat{f(a)} = f(\hat{a}) = \hat{a}$. By Remark 1.3.14.1 we have $\hat{a} = a$. Thus $a = f(a)$. A contradiction. □

[12] This can also be found as Lemma 4 in [Jech, 1971, p.7].

Theorem 1.3.20. *Any set of ordinals is well-ordered by* $\in$.[13]

Proof. We show that any non-empty set of ordinals has a least element. Let $\mathfrak{O} \neq \emptyset$ be a set of ordinal numbers. Pick any $\mathfrak{b} \in \mathfrak{O}$. Note that $\mathfrak{O}' := \{\mathfrak{a} \in \mathfrak{O} \mid \mathfrak{a} \leqslant \mathfrak{b}\} \neq \emptyset$ has a least element if and only if $\mathfrak{O}$ has one. By Lemma 1.3.19 we see that $\mathfrak{O}' \subseteq \mathfrak{b}$. As $\mathfrak{b}$ is well-ordered we obtain the existence of a least element $\mathfrak{a} \in \mathfrak{O}' \subseteq \mathfrak{O}$. □

1.3.2 Induction on Well-Orderings

Let $(P, <)$ be any well-ordered set. Let $\mathsf{Predicate}$ be any predicate on elements of P, i.e. for $p \in P$ the statement $\mathsf{Predicate}(p)$ is either true or false.

Theorem 1.3.21. *We have* $\forall p \in P : \mathsf{Predicate}(p)$ *if the following two conditions hold*

(a) We have $\mathsf{Predicate}(p_0)$ *where* p_0 *is the least element of* P.

(b) If $\mathsf{Predicate}(q)$ *for all* $q < p$, *then* $\mathsf{Predicate}(p)$.

Remark 1.3.21.1. As p_0 has no predecessors and the universal quantification over an empty set always returns a true statement *(b)* implies *(a)*. So we could also get rid of the first condition.

Proof. Let $W := \{q \in P \mid \neg\mathsf{Predicate}(q)\}$. Assume that W is non-empty. Then W has a least element p. By *(a)* we know that $p \neq p_0$. As p is the least element of W, we have $\forall q < p : \mathsf{Predicate}(q)$. By *(b)* we see, however, that $p \notin W$, a contradiction. □

Remark 1.3.21.2. i) The main idea of this form of induction is that there can be no least witness of the falsehood of $\mathsf{Predicate}$. And if there cannot be a least witness there can be no witness. Which in turn means that $\mathsf{Predicate}$ must hold everywhere.

ii) Clearly we have for $A \subseteq P$ that $A = P$ whenever $p_0 \in A$ and $p \in A$ whenever $q \in A$ for all $q < p$.

Lemma 1.3.22. *The union of any set of ordinals is again an ordinal.*[14]

Proof. Let I be a set and for $i \in I$ let $\mathfrak{a}_i$ be an ordinal. Define $\mathfrak{a} := \bigcup_{i \in I} \mathfrak{a}_i$. In order to see that $(\mathfrak{a}, \in)$ is well-ordered pick any non-empty subset $X \subseteq \mathfrak{a}$. Pick $x \in X$ and choose $i \in I$ such that $x \in \mathfrak{a}_i$. Then clearly $X' = \{y \in X \mid y \leqslant x\} \subseteq \mathfrak{a}_i$ has a least element z. Clearly z is also a least element of X. In order to see that $\mathfrak{a}$ is a transitive set let $x \in \mathfrak{a}$. There is $i \in I$ such that $x \in \mathfrak{a}_i$. As $\mathfrak{a}_i$ is a transitive set we have $x \subseteq \mathfrak{a}_i$. Thus also $x \subseteq \mathfrak{a}$. □

Lemma 1.3.23. *There is at most one isomorphism between well-ordered sets.*

We already saw that this follows from Lemma 1.3.16. Now we give a proof via induction:

[13] To be found in [Jech, 1971, p.8].

[14] This is also Lemma 5 in [Jech, 1971, p.8].

Proof. Suppose that $(P, <)$ and $(Q, \prec)$ are well-ordered. Let f and g be two isomorphisms $(P, <) \to (Q, \prec)$. Clearly f and g both must send p_0, the least element of P, to the least element of Q. Thus $f(p_0) = g(p_0)$. Now let $p \in P$ and suppose that for all $q < p$ we have $f(q) = g(q)$. Then p must be send by f to the least element greater than any element of $\{f(q) \mid q < p\}$. Similarly g sends p to the least element greater than any element of $\{g(q) \mid q < p\}$. Obviously thus $g(p) = f(p)$. By Theorem 1.3.21 we have $f = g$. □

Theorem 1.3.24. *Every well-ordered set is isomorphic to some (unique) ordinal.*[15]

Proof. Let $(P, <)$ be well-ordered. Consider $(\hat{P} := \{\hat{p} \mid p \in P\}, \subseteq)$, which is isomorphic to $(P, <)$. We proceed by induction on P. Clearly the ordinal $\emptyset$ is isomorphic to $\emptyset \in \hat{P}$. Now suppose that for $p \in P$ we have that for any $q < p$ the initial segment $\hat{q}$ is isomorphic to some ordinal $\mathfrak{a}_q$. We consider two cases: Either p is the least element greater than some $q < p$ or it is not. If it is we see that $\hat{p}$ is just $\hat{q} \cup \{q\}$. Thus the ordinal $\mathfrak{a}_p := \mathfrak{a}_q \cup \{\mathfrak{a}_q\}$ is isomorphic to $\hat{p}$. Now assume that it is not. By Lemma 1.3.22 $\mathfrak{a} := \bigcup_{q<p} \mathfrak{a}_q$ is an ordinal. Let f_q be the isomorphism between $\mathfrak{a}_q$ and $\hat{q}$. By Lemma 1.3.23 we conclude that

$$f_p : \mathfrak{a} \longrightarrow \hat{p},\ a \longmapsto \begin{cases} f_q(a) & a \in \mathfrak{a}_q \end{cases}$$

is well-defined. Clearly f_p is order-preserving and injective. Its range is $A := \bigcup_{q<p} \hat{q}$. As p is not the least element greater than some $q < p$ we see that for any $a < p$ there is $q < p$ such that $a < q$. Thus $A = \hat{p}$ and f_p is an isomorphism. So we have an isomorphism $f_p : \mathfrak{a}_p \to \hat{p}$ for any $p \in P$.

As P was arbitrary the same works for $P+1$ where we add a new element $P_\infty \notin P$, i.e. $P + 1 = P \cup \{P_\infty\}$ and set $p < P_\infty$ for any $p \in P$. Note that $\hat{P}_\infty = P$ and thus $(\mathfrak{a}_{P_\infty}, \in)$ is isomorphic to $(P, <)$. □

1.3.3 Limit Ordinals and Induction on Ordinals

In the proof of Theorem 1.3.24 we had to differentiate two cases. Either our element was the direct successor of a previous element or not. This distinction is also relevant for ordinals.

Definition 1.3.25 (Successor Ordinal)**.** For any ordinal $\mathfrak{a}$ we define its **successor**

$$\mathfrak{a} + 1 := \mathfrak{a} \cup \{\mathfrak{a}\}\ .$$

Remark 1.3.25.1. $\mathfrak{a} + 1$ is the least ordinal greater than $\mathfrak{a}$.

Definition 1.3.26 (Limit ordinal)**.** We call an ordinal $\mathfrak{a}$ a **limit ordinal** if and only if $\mathfrak{a} > \mathfrak{b} + 1$ for any $\mathfrak{b} < \mathfrak{a}$.

Remark 1.3.26.1. The ordinal ω isomorphic to the well-ordered set $(\mathbb{N}, <)$ is the paradigmatic example of a limit ordinal.

With this distinction at hand we can proof the following

[15] This is also found as Theorem 1 in [Jech, 1971, p.8].

Theorem 1.3.27 (Transfinite Induction)**.** *Let $\mathfrak{O}$ be a class of ordinals. $\mathfrak{O}$ contains all ordinals if and only if the three conditions hold*

(i) $\emptyset \in \mathfrak{O}$.

(ii) If $\mathfrak{a} \in \mathfrak{O}$ *then* $\mathfrak{a} + 1 \in \mathfrak{O}$

(iii) If $\mathfrak{a}$ *is a limit ordinal and for any* $\mathfrak{b} < \mathfrak{a}$ *we have* $\mathfrak{b} \in \mathfrak{O}$ *then* $\mathfrak{a} \in \mathfrak{O}$.[16]

Proof. Note that *(ii)* and *(iii)* together are just equivalent to *(b)* from Theorem 1.3.21. Now let $\mathfrak{a}$ be any ordinal. Let $P := \{\mathfrak{b} \mid \mathfrak{b} \leqslant \mathfrak{a}\}$. By Theorem 1.3.21 we see that $\mathfrak{O} \cap P = P$. So $\mathfrak{a} \in \mathfrak{O}$. □

Theorem 1.3.28 (Transfinite Recursion)**.** *Let G be a function defined on the class of all sets mapping into the class of all sets. There is a function F defined on the class of all ordinals mapping into the class of all sets such that for every ordinal $\mathfrak{a}$ we have*

$$F(\mathfrak{a}) = G(F|_{\mathfrak{a}})\,. \tag{1}$$

F is uniquely determined by (1).[17]

Proof. Consider the class of partially defined functions which satisfy (1) whenever defined, i.e.

$$C := \{f \text{ function} \mid \operatorname{dom}(f) \text{ is an ordinal and } \forall \mathfrak{b} \in \operatorname{dom}(f) : F(\mathfrak{b}) = G(F|_{\mathfrak{b}})\}\,.$$

Now let $f, g \in C$ such that $\operatorname{dom}(g) = \operatorname{dom}(f)$. Suppose $f \neq g$. Then there is a least ordinal $\mathfrak{a}$ such that $g(\mathfrak{a}) \neq f(\mathfrak{a})$. Then $f|_{\mathfrak{a}} = g|_{\mathfrak{a}}$ and thus

$$g(\mathfrak{a}) = G(g|_{\mathfrak{a}}) = G(f|_{\mathfrak{a}}) = f(\mathfrak{a})\,. \tag{2}$$

A contradiction. Further, we can conclude that C is linearly ordered by

$$f < g :\Leftrightarrow \operatorname{dom}(f) \subset \operatorname{dom}(g)\,.$$

This means that $F = \bigcup C$ is a function. Clearly $F \in C$. Note that

$$\operatorname{dom}(F) = \bigcup \{\operatorname{dom}(f) \mid f \in C\}\,.$$

Suppose that F is not defined for all ordinals. Then $\operatorname{dom}(F)$ is again an ordinal. Let $\mathfrak{a}$ be the least ordinal for which F is not defined. Define a function F^* by $F^*(\mathfrak{a}) = G(F)$ and $F^*(\mathfrak{b}) = F(\mathfrak{b})$ for $\mathfrak{b} < \mathfrak{a}$. Then $F^* \in C$ and $\operatorname{dom}(F^*) \supset \operatorname{dom}(F)$. A contradiction. Thus F is defined on all ordinals. Equation (2) shows that by Transfinite Induction 1.3.27 (1) determines F uniquely. □

[16] This is Theorem 2 in [Jech, 1971, p.8].
[17] This is Theorem 3 in [Jech, 1971, p.9].

Remark 1.3.28.1. This theorem shows that we can define a function on all ordinal numbers by recursion. Recursion means to declare how $F(\mathfrak{a})$ shall be defined whenever $F(\mathfrak{b})$ for all $\mathfrak{b} < \mathfrak{a}$ is already defined. This recursion is modelled as the function G in the theorem. The formula $F(\mathfrak{a}) = G(F|_{\mathfrak{a}})$ is to be understood as: The value of F at the ordinal $\mathfrak{a}$ depends in a manner described by G on the values on $\mathfrak{b} < \mathfrak{a}$ as given in the restricted function $F|_{\mathfrak{a}}$. The projection to the first coordinate of $F|_{\mathfrak{a}}$ is $\mathfrak{a}$. So G has information about in which step of the recursion we are. In particular, the value of G can depend on the step of the recursion.

When it is clear from the context how G must be defined, we will not construct G in order to apply the theorem.

Note that for any finite number $n \in \mathbb{N}$ there is exactly one ordinal $\mathfrak{a}_n$ of that cardinality. Notationally we will often write n instead of $\mathfrak{a}_n$.

1.3.4 Cardinal Numbers

Now we consider sets without any additional structure.

First we just define a *"formal"* notation of cardinality which we will later fill with the *"material"* of cardinal numbers.

Definition 1.3.29. Given two sets X and Y we will write

$\mathsf{card}(X) \leqslant \mathsf{card}(Y)$ if and only if there is an injection $f : X \to Y$.

$\mathsf{card}(X) = \mathsf{card}(Y)$ if and only if there is a bijection $f : X \to Y$.

$\mathsf{card}(X) < \mathsf{card}(Y)$ if and only if $\mathsf{card}(X) \leqslant \mathsf{card}(Y)$ but not $\mathsf{card}(X) = \mathsf{card}(Y)$.

Theorem 1.3.30 (Cantor)**.** *For any set X we have $\mathsf{card}(X) < \mathsf{card}(\mathfrak{P}(X))$.*[18]

Proof. Clearly $f : X \to \mathfrak{P}(X),\ x \mapsto \{x\}$ is an injection. Now suppose that there is a bijection $g : X \to \mathfrak{P}(X)$. Define $A := \{x \in X \mid x \notin g(x)\} \in \mathfrak{P}(X)$. Now as g is surjective there is $a \in X$ such that $g(a) = A$. Suppose $a \in g(a)$. Then $a \notin A$ by the definition of A. This is a contradiction. Conversely suppose $a \notin g(a)$. Then $a \in A$ by the definition of A. This is a contradiction. Both cases fail so there cannot be a bijection g. □

Theorem 1.3.31. *The relation $X \prec Y :\Leftrightarrow \mathsf{card}(X) < \mathsf{card}(Y)$ is a partial ordering.*

Proof. Obviously $X \nprec X$ as Id_X is a bijection $X \to X$. Now $\prec$ is transitive as the composition of injective functions is still injective. □

Remark 1.3.31.1. Clearly $\prec$ is not a total order as for example for $a \neq b$ the sets $\{a\}$ and $\{b\}$ can not be in relation as they are bijective but not equal. One can however hope that the order is total up to bijectivity. In fact:

Theorem 1.3.32. *Cardinality orders the class of all sets totally up to bijectivity if and only if the* AXIOM OF CHOICE *holds.*

[18]This is Theorem 5 in [Jech, 1971, p.10].

Proof. Recall that the AXIOM OF CHOICE (AC) is equivalent to the WELL-ORDERING PRINCIPLE which states that for every set P there is an order $<$ such that $(P, <)$ is well-ordered.

First assume AC. Let X and Y be non-bijective. Let $<$ and $\prec$ be such that $(X, <)$ and $(Y, \prec)$ are well-ordered. By Theorem 1.3.24 there are ordinals $\mathfrak{a}$ and $\mathfrak{b}$ such that there are isomorphisms

$$\Psi_{\mathfrak{a}} : (\mathfrak{a}, \in) \longrightarrow (X, <)$$
$$\Psi_{\mathfrak{b}} : (\mathfrak{b}, \in) \longrightarrow (Y, \prec)$$

By Lemma 1.3.17 either $\mathfrak{a}$ is isomorphic to an initial segment of $\mathfrak{b}$ or vice versa. Say there is f and $b \in \mathfrak{b}$ such that $f : \mathfrak{a} \to \hat{b}$ is a bijection. Then $f : \mathfrak{a} \to \mathfrak{b}$ is an injection. And $\Psi_{\mathfrak{b}}^{-1} \circ f \circ \Psi_{\mathfrak{a}} : X \to Y$ is an injection.

Now assume that $\prec$ is total up to bijectivity. Let X be a set. If X is countable then it is well-orderable so assume X is uncountable. Consider the set $\mathfrak{O} := \{\mathfrak{a} \text{ ordinal} \mid \mathsf{card}(\mathfrak{a}) < \mathsf{card}(X)\}$. Then $\mathfrak{b} := \bigcup \mathfrak{O}$ is an ordinal and infinite as ω is an infinite cardinal with cardinality less than X. Suppose that $\mathfrak{b} \in \mathfrak{O}$, then $\mathfrak{b}+1 \in \mathfrak{O}$ as $\mathsf{card}(\mathfrak{b}) = \mathsf{card}(\mathfrak{b}+1)$. This is a contradiction as $\mathfrak{b}$ is the supremum of $\mathfrak{O}$. So $\mathfrak{b} \notin \mathfrak{O}$. As $\prec$ is total this means that $\mathsf{card}(\mathfrak{b}) \geqslant \mathsf{card}(X)$. In particular there is an injection $f : X \to \mathfrak{b}$. We define $x < y :\Leftrightarrow f(x) \in f(y)$ and see that $(X, <)$ is isomorphic to $f(X) \subseteq \mathfrak{b}$ and thus well-ordered. We have proven that any set is well-orderable and thus have proven AC. □

Definition 1.3.33 (Cardinal Number)**.** An ordinal $\mathfrak{a}$ is called **cardinal number** if and only if no $\mathfrak{b} < \mathfrak{a}$ is bijective to $\mathfrak{a}$.

Remark 1.3.33.1. Cardinal numbers are in a vivid way the smallest ordinals of a given size.

Definition 1.3.34 (The Aleph-Notation)**.** We define by Transfinite Recursion 1.3.28 $\aleph_0 = \omega$, $\aleph_{\mathfrak{a}+1} = \min\{\mathfrak{b} \text{ cardinal} \mid \aleph_{\mathfrak{a}} < \mathfrak{b}\}$ and for a limit ordinal $\mathfrak{b}$ we define $\aleph_{\mathfrak{b}} := \bigcup\{\aleph_{\mathfrak{a}} \mid \mathfrak{a} < \mathfrak{b}\}$.

Remark 1.3.34.1. The aleph notation yields a bijection between the ordinals and the cardinals.

Theorem 1.3.35. *If we assume the* AXIOM OF CHOICE *then for any set X there is a cardinality $\mathfrak{k}_X$ such that* $\mathsf{card}(X) = \mathsf{card}(\mathfrak{k}_X)$.

Proof. By AC we can well-order X. By Theorem 1.3.24 there is an ordinal isomorphic to that well-ordered set. Thus the set K of ordinals with the same cardinality as X is non-empty. Let $\mathfrak{k}_X$ be the least element of K. Clearly $\mathfrak{k}_X$ is a cardinality. □

Now we can turn the formal notation $\mathsf{card}(X)$ which until now only made sense in statements like $\mathsf{card}(X) < \mathsf{card}(Y)$ into an object by defining $\mathsf{card}(X) = \mathfrak{k}_X$.

Even without the assumption of the AXIOM OF CHOICE the cardinality of ordinals is unbounded:

Proposition 1.3.36. *For each ordinal there is another ordinal of greater cardinality.*

Proof. Let $\mathfrak{b}$ be at least countably infinite. The set

$$\mathfrak{O} := \{\mathfrak{a} \text{ ordinal} \mid \mathsf{card}(\mathfrak{a}) \leqslant \mathsf{card}(\mathfrak{b})\}$$

is non-empty. Now let $\mathfrak{o} := \bigcup \mathfrak{O}$. If $\mathfrak{o} \in \mathfrak{O}$ then $\mathfrak{o} + 1 \in \mathfrak{O}$ as $\mathsf{card}(\mathfrak{o}) = \mathsf{card}(\mathfrak{o} + 1)$. But as $\mathfrak{o}$ is the supremum of $\mathfrak{O}$ this is impossible, so $\mathfrak{o} \notin \mathfrak{O}$. As ordinals are totally ordered we conclude from $\mathfrak{o} > \mathfrak{b}$ that there is an injection from $\mathfrak{b}$ to $\mathfrak{o}$. However as $\mathfrak{o} \notin \mathfrak{O}$ there is no bijection. Thus $\mathsf{card}(\mathfrak{o}) > \mathsf{card}(\mathfrak{b})$. □

1.4 Algebraic Tools

Definition 1.4.1 (Group Action)**.** Let X be any set and G a group. A function $\alpha : G \times X \to X$ is called a **group action** if and only if

$$\forall h, g \in G : \forall x \in X : \alpha(h, \alpha(g, x)) = \alpha(hg, x)\,.$$

We call α an **anti-action** if and only if

$$\forall h, g \in G : \forall x \in X : \alpha(h, \alpha(g, x)) = \alpha(gh, x)\,.$$

Definition 1.4.2 (Algebraical Transitivity)**.** Let X be any set and G a group. A group action $\alpha : G \times X \to X$ is called **algebraically transitive** if and only if

$$\forall x, y \in X : \exists g \in G : \alpha(g, x) = y\,.$$

Sometimes we need a more general notion than that of a group.

Definition 1.4.3 (Semigroup)**.** Let S be a set and $\cdot : S \times S \to S$ an operation. We call $(S, \cdot)$ a **semigroup** if and only if $\cdot$ is associative i.e.

$$\forall a, b, c \in S : (a \cdot b) \cdot c = a \cdot b \cdot c = a \cdot (b \cdot c)\,.$$

Definition 1.4.4 (Super- / Subadditivity)**.** Let $(H, *)$ be a semigroup. A map $a : H \to \mathbb{R}$ is called **superadditive** if and only if we have $a(h * j) \geqslant a(h) + a(j)$ for all $h, j \in H$. a is called **subadditve** if and only if we have $a(h * j) \leqslant a(h) + a(j)$ for all $h, j \in H$.

Definition 1.4.5 (Monoid)**.** Let M be a set and $\cdot : M \times M \to M$ an operation. We call $(M, \cdot)$ a **monoid** if and only if $\cdot$ is associative and there is an identity $e \in M$, i.e.

$$\forall m \in M : e \cdot m = m = m \cdot e\,.$$

Remark 1.4.5.1.
1. Just as with groups we will often suppress the operation notationally with monoids and semigroups. So we often call S itself a semigroup or M itself a monoid. Also we will sometimes denote $ab = a \cdot b$.
2. With the operation $\circ$ the set $\mathcal{F}(X, X)$ becomes a monoid.

Definition 1.4.6 (Monoid Homomorphism)**.** Let $(M, \cdot)$ and $(N, *)$ be two monoids. A map $f : M \to N$ is called a **monoid homomorphism** if and only if $f(a \cdot b) = f(a) * f(b)$ for any $a, b \in M$.

The class of all monoids together with the monoid homomorphisms forms the category Mon.

Definition 1.4.7 (Monoid Action)**.** Let X be any set and $(M, \cdot)$ a monoid. A function $\alpha : M \times X \to X$ is called a **monoid action** if and only if

$$\forall m, n \in M : \forall x \in X : \alpha(m, \alpha(n, x)) = \alpha(m \cdot n, x)\,.$$

Remark 1.4.7.1. A monoid action α of M on X induces a monoid homomorphism $M \to \mathcal{F}(X, X)$, $m \mapsto \alpha(m, \cdot)$.

Definition 1.4.8 (Faithful)**.** A monoid action α is called faithful if the monoid homomorphism $m \mapsto \alpha(m, \cdot)$ has trivial kernel.

1.5 Analytical Tools

1.5.1 Function Spaces

Let X and Y be locally compact HAUSDORFF spaces and $\Delta_X := \{(x, x) \mid x \in X\} \subseteq X \times X$ and $\Delta_Y \subseteq Y \times Y$ be the diagonals. By $\mathcal{C}(X, Y)$ we denote the set of continuous functions from X to Y.

Definition 1.5.1 (Equicontinuity for CHaus)**.** A family $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is called **equicontinuous** if and only if

$$\forall U \in \mathcal{U}(\Delta_Y) : \exists V \in \mathcal{U}(\Delta_X) : \forall f \in \mathcal{F} : (f \times f)(V) \subseteq U\,.$$

Theorem 1.5.2 (Arzelà-Ascoli)**.** *Let $\mathcal{C}(X, Y)$ be equipped with the topology of uniform convergence.*[19] *Then $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is compact if and only if*[20]

(a) $\mathcal{F}$ is closed.

(b) $\mathcal{F}$ is equicontinuous.[21]

Theorem 1.5.3 (Dini's Theorem)**.** *Let $(f_i)_{i \in I} \in \mathcal{C}(X, \mathbb{R})^I$ be a net for some directed set $(I, <)$. If $(f_i)_{i \in I}$ converges monotonically and pointwise to a continuous function $f \in \mathcal{C}(X, \mathbb{R})$, then $(f_i)_{i \in I}$ converges uniformly to f.*[22]

Proof. Assume $(f_i)_{i \in I}$ is decreasing and let $C := \sup_{x \in X} f_{i_0}$ for some $i_0 \in I$. W.l.o.g. i_0 is a least element of I. We consider the space $X \times \mathbb{R}$ with the product topology. For $g, h \in \mathcal{C}(X, \mathbb{R})$ let

$$[g, h] := \{(x, y) \in X \times \mathbb{R} \mid g(x) \leqslant y \leqslant h(x)\}\,.$$

Clearly, $[f, f_i]$ is closed. As $f_i \xrightarrow{i \in I} f$ we have $\bigcap_{i \in I} [f, f_i] = \{f\}$. If $i < j$ then $f_i \leqslant f_j$ and so $[f, f_i] \supseteq [f, f_j]$. Hence, $[f, f_i]^c$ is increasing. Let $\varepsilon > 0$. Clearly $\{[f, f_i]^c \mid i \in I\}$ is an open cover of

$$K_\varepsilon := [f, C] \setminus \left\{(x, y) \in X \times \mathbb{R} \;\middle|\; f(x) < y < f(x) + \frac{\varepsilon}{2}\right\}\,.$$

[19] If X is merely **locally** compact, this needs to be the topology of uniform convergence **on compact sets**.

[20] If Y is not compact HAUSDORFF but just a uniform space one needs to add the third condition that $\mathcal{F}[x] := \{f(x) \mid f \in \mathcal{F}\}$ has compact closure for each $x \in X$.

[21] This is Theorem 17 of Chapter 7 in [Kelley, 1975, p.233 et seq.].

[22] Theorem and proof can be found as an exercise in [Kelley, 1975, p.239].

As X is compact we see that K_ε is compact. Thus, there is a finite set $\{i_1, \dots i_n\}$ such that $\bigcup_{j=1}^n [f, f_{i_j}]^c \supseteq K_\varepsilon$. Define $J := \max(i_1, \dots, i_n)$. As $([f, f_i]^c)_{i \in I}$ is increasing we have $[f, f_j]^c \supseteq K_\varepsilon$ for all $j > J$. Thus $f \leqslant f_j < f + \varepsilon$ for all $j > J$. This obviously implies uniform convergence. □

1.5.2 Multivalued Functions

Here I follow closely and reuse parts of the presentation in my bachelor's thesis [Haupt, 2020, p.10 et seq.].

Definition 1.5.4 (Multivalued Function)**.** Let X and Y be sets. A **multivalued function** between X and Y is a function $\varphi : X \to \mathfrak{P}(Y)$. We write $\varphi : X \twoheadrightarrow Y$ to denote multivalued functions.

Let $\varphi : X \twoheadrightarrow Y$ be a multivalued function.

Definition 1.5.5 (Upper Preimage)**.** Let $A \subseteq Y$ be given. The **upper preimage** $\varphi^u(A)$ of A is given by

$$\varphi^u(A) := \left\{ x \in X \;\middle|\; \varphi(x) \subseteq A \right\}$$

If X and Y are equipped with topologies τ_X and τ_Y, one can define a notion of continuity for multivalued functions.

Definition 1.5.6 (Upper Hemi-Continuity)**.** φ is called **upper hemi-continuous**, if and only if the upper preimages of open sets are open, i.e. $\varphi^u(\tau_Y) \subseteq \tau_X$.

Definition 1.5.7 (Closed-Valued)**.** φ is called **closed-valued** if and only if the set $\varphi(x) \subseteq Y$ is closed for any $x \in X$.

Definition 1.5.8 (Graph)**.** The **graph** of a multivalued function $\varphi : X \twoheadrightarrow Y$ is given by

$$\operatorname{Graph}(\varphi) := \left\{ (x, y) \in X \times Y \;\middle|\; y \in \varphi(x) \right\} .$$

For a (non-multivalued) $f : X \to Y$ the graph is given by

$$\operatorname{Graph}(f) := \left\{ (x, y) \in X \times Y \;\middle|\; y = f(x) \right\} .$$

Theorem 1.5.9 (Closed Graph Theorem for Multivalued Functions)**.** *For a multivalued function $\varphi : X \twoheadrightarrow Y$ into a compact* HAUSDORFF *space Y, the following are equivalent:*

a) $\operatorname{Graph}(\varphi)$ *is closed (in the product topology).*

b) φ is closed-valued and upper hemi-continuous.[23]

[23] This is Theorem 17.11. in [Border, 2006].

Corollary 1.5.9.1. *Let $f : X \to Y$ be continuous. Then*

$$f^{-1} : Y \twoheadrightarrow X,\ y \longmapsto f^{-1}(\{y\})$$

is upper hemi-continuous.

Proof. As f is continuous Graph(f) is closed. Note that

$$s : X \times Y \longrightarrow Y \times X,\ (x, y) \longmapsto (y, x)$$

is a homeomorphism with respect to the product topologies. Thus $s(\text{Graph}(f))$ is closed. Now note that $s(\text{Graph}(f)) = \text{Graph}(f^{-1})$. □

When (X, d) is a metric space we will denote by $B_\varepsilon(x)$ the open ball of radius ε around x, i.e.

$$B_\varepsilon(x) := \{x' \in X \mid d(x, x') < \varepsilon\} .$$

Lemma 1.5.10. *Let A and B be metric spaces and $\Psi : A \twoheadrightarrow B$ an upper hemi-continuous multi-valued function. Then*

$$\mathsf{diam} \circ \Psi : A \longrightarrow \mathbb{R},\ a \longmapsto \mathsf{diam}(\Psi(a))$$

is an upper semi-continuous real-valued function.

Proof. The upper hemi-continuity of Ψ implies that the set of all y' such that $\Psi(y') \subseteq B_\varepsilon(\Psi(\{y\}))$ is an open neighbourhood of y. However this means that the set of all y' such that $\mathsf{diam}(\Psi(y')) < \mathsf{diam}(\Psi(y)) + 2\varepsilon$ is a neighbourhood around y. This exactly is the desired upper semi-continuity. □

1.5.3 Degree of Denseness

Let (X, d) be a metric space. For any subset $A \subseteq X$ we denote

$$B_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x) .$$

Definition 1.5.11. Let $A \subseteq X$. We define the **degree of denseness** of A as

$$\text{denseness}(A) = \inf \{\varepsilon > 0 \mid B_\varepsilon(A) = X\} .$$

Proposition 1.5.12. *Let (X, G, α) be a dynamical system and $K \subseteq G$ compact. Then the mapping*

$$\kappa_K : X \longrightarrow \mathbb{R}_0^+,\ x \longmapsto \sup \{d(y, \alpha(K, x)) \mid y \in X\} = \text{denseness}(\alpha(K, x))$$

is continuous.

Proof. Firstly, we prove that κ_K is lower semi-continuous. The graph of $\alpha(K, \cdot) : x \mapsto \alpha(K, x)$ is closed: Let $(x_n, y_n)_{n\in\mathbb{N}} \in \text{Graph}(\alpha(K, \cdot))^{\mathbb{N}}$ converge to $(x, y) \in X \times X$. For $n \in \mathbb{N}$ there is $k_n \in K$ such that $\alpha(k_n, x_n) = y_n$. Pick a converging subsequence of $(k_n)_{n\in\mathbb{N}}$ with limit $k \in K$. Then $\alpha(k, x) = y$ by the (joint) continuity of $\alpha : G \times X \to X$. We conclude $(x, y) \in \text{Graph}(\alpha(K, \cdot))$. So by Theorem 1.5.9

$\alpha(K,\cdot)$ is upper hemi-continuous. If $\kappa_K(x) = 0$ then the value at x is minimal, thus x is trivially a point of lower semi-continuity. Now assume $\kappa_K(x) > 0$. Let $\varepsilon : 0 < \varepsilon < \kappa_K(x)$ be arbitrary and define $A := \alpha(K,x)$. Fix $\rho > 0$. Consider the set

$$U_\rho := \{x' \in X \mid \alpha(K,x') \subseteq B_\rho(\alpha(K,x))\}$$

For any $x' \in U_\rho$ let $D = \alpha(K,x')$. Then

$$B_{\varepsilon-\rho}(B) \subseteq B_{\varepsilon-\rho}(B_\rho(A)) \subseteq B_\varepsilon(A) \subset X\,.$$

Letting $\varepsilon \to \kappa_K(x)$ we conclude $\kappa_K(x') = \text{denseness}(B) \geqslant \kappa_K(x) - \rho$ for any $x' \in U_\rho$. Furthermore, U_ρ is an open neighbourhood of x by the upper hemi-continuity of $\alpha(K,\cdot)$. This shows the lower semi-continuity of κ_K.

Secondly, we prove the upper semi-continuity of κ_K. By compactness of K the family $\{\alpha(k,\cdot) \mid k \in K\}$ is (uniformly) equicontinuous. Fix $x \in X$ and let $\delta > 0$ be arbitrary. Define $\beta := \delta + \kappa_K(x)$. For any $y \in X$ there is $k_y \in K$ such that $\alpha(k_y, x) \in B_\delta(y)$ as $\beta > \kappa_K(x)$. By equicontinuity there is an $\eta > 0$ such that

$$\forall x' \in B_\eta(x) : \forall k \in K : d(\alpha(k,x), \alpha(k,x')) < \delta\,.$$

So we have $\alpha(k_y, x') \in B_\delta(B_\beta(y))$ for any $x' \in B_\eta(x)$. This yields that

$$\forall y \in X : \delta + \beta \geqslant d(y, \alpha(K,x'))$$

for any $x' \in B_\eta(x)$. Note that $\kappa_K(x') = \sup_{y\in Y} d(y, \alpha(K,x'))$. Hence, $2\delta + \kappa_K(x) = \delta + \beta \geqslant \kappa_K(x')$ in a neighbourhood of x. So κ_K is upper semi-continuous. □

Proposition 1.5.13. *Let $\mathbf{X} = (X, G, \alpha)$ be a minimal topological dynamical system. Let $(K_n)_{n\in\mathbb{N}} \in \mathfrak{P}(G)$ be an increasing sequence of subsets of the group G such that $\bigcup_{n\in\mathbb{N}} K_n = G$. Then $(\kappa_{K_n})_{n\in\mathbb{N}}$ converges monotonically and pointwise to zero.*

Proof. Let $\varepsilon > 0$ and $x \in X$. Choose finitely many $x_1, \dots, x_n \in X$ such that

$$\bigcup_{i=1}^{n} B_{\frac{\varepsilon}{2}}(x_i) = X\,.$$

As $\mathbf{X}$ is minimal $\alpha(G,x)$ is dense. For each $i \in \{1,\dots,n\}$ there is $g_i \in G$ such that $\alpha(g_i,x) \in B_{\frac{\varepsilon}{2}}(x_i)$. As $\bigcup_{n\in\mathbb{N}} K_n = G$ there is $N \in \mathbb{N}$ such that $\{g_1,\dots,g_n\} \subseteq \bigcup_{i=1}^{N} K_i$. As the sequence $(K_n)_{n\in\mathbb{N}}$ is increasing we have $\{g_1,\dots,g_n\} \subseteq K_n$ for any $n > N$. Thus $\kappa_{K_n}(x) \leqslant \varepsilon$. This shows the pointwise convergence. The fact that the convergence is monotone follows from the monotonicity of $(K_n)_{n\in\mathbb{N}}$. □

Corollary 1.5.13.1. *If the K_n are compact then by* DINI*'s Theorem 1.5.3 and Proposition 1.5.12 the sequence $(\kappa_{K_n})_{n\in\mathbb{N}}$ converges uniformly to zero.*

Remark 1.5.13.1. Note that if $\mathbf{X}$ is not minimal this Proposition fails to hold, has $\kappa_G(x) > 0$ for any x which has no dense orbit.

1.6 Measure Theory

Let X and Y be sets.

1.6.1 Notation

Let $\mathfrak{E} \subseteq \mathfrak{P}(Y)$.

Definition 1.6.1 (Generated σ-algebra)**.** We define the σ-algebra **generated** by $\mathfrak{E}$ as

$$\sigma(\mathfrak{E}) := \bigcap_{\substack{\mathfrak{A} \supseteq \mathfrak{E} \\ \mathfrak{A} \text{ is a } \sigma\text{-algebra}}} \mathfrak{A}\,.$$

Lemma 1.6.2 (Generator Theorem)**.** *Let $f : X \to Y$ be any function. Then the following equality holds*

$$\sigma\left(f^{-1}(\mathfrak{E})\right) = f^{-1}(\sigma(\mathfrak{E}))\ .$$

Let $(X_i, \mathfrak{A}_i, \mu_i)_{i\in I}$ be a family of measure spaces. The system of *"boxes"* given by the $\mathfrak{A}_i$ is denoted as

$$*_{i\in I}\,\mathfrak{A}_i := \left\{ \prod_{i\in I} A_i \;\middle|\; A_i \in \mathfrak{A}_i \text{ and } \mathsf{card}\left(\{i \in I \mid A_i \neq X_i\}\right) \leqslant \aleph_0 \right\}\ .$$

The **product σ-algebra** is then denoted by

$$\bigotimes_{i\in I} \mathfrak{A}_i := \sigma\left(*_{i\in I}\,\mathfrak{A}_i\right)\ .$$

The product measure is denoted by $\bigotimes_{i\in I} \mu_i$.

Recall the notion of a Dynkin-System.

Definition 1.6.3 (Dynkin System)**.** We call $\mathfrak{D} \subseteq \mathfrak{P}(X)$ a **Dynkin system** if and only if

1. $\emptyset \in \mathfrak{D}$
2. For any $D \in \mathfrak{D}$ we have $D^c \in \mathfrak{D}$.
3. For any $(D_n)_{n\in\mathbb{N}} \in \mathfrak{D}^{\mathbb{N}}$ pairwise disjoint we have

$$\biguplus_{n\in\mathbb{N}} D_n \in \mathfrak{D}$$

Dynkin systems are also called λ-systems.

For ease of notation we use the

Definition 1.6.4 (π-system)**.** A subset $\mathfrak{H} \subseteq \mathfrak{P}(X)$ is called **intersection stable** if and only if

$$\forall A, B \in \mathfrak{H} : A \cap B \in \mathfrak{H}\,.$$

An intersection stable set is also called a **π-system**.

Dynkin systems are related to σ-algebras by the

Theorem 1.6.5 (Dynkin-π-λ Theorem)**.** *Let $\mathfrak{E} \subseteq \mathfrak{P}(Y)$ be a π-system. The λ-system generated by $\mathfrak{E}$ is a σ-algebra.*[24]

If the σ-algebras are clear from the context we denote by $\mathcal{M}(X, Y)$ the family of measurable functions between measurable spaces $(X, \mathfrak{A})$ and $(Y, \mathfrak{F})$.

[24] This is Theorem I.6.7 in [Elstrodt, 2011, p.25].

1.6.2 Measures

Let $(X, \mathfrak{A})$ be a measurable space and μ a measure.

Definition 1.6.6 (Probability)**.** We call μ a **probability** if and only if $\mu(X) = 1$.

Definition 1.6.7 (Dirac Measure)**.** Let $x \in X$. The **Dirac Measure** in x is the probability measure

$$\delta_x : \mathfrak{A} \longrightarrow [0,1],\ A \longmapsto \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Definition 1.6.8 (Pushforward Measure)**.** Let $f : (X, \mathfrak{A}) \to (Y, \mathfrak{C})$ be a measurable function. The **pushforward measure** on $(Y, \mathfrak{C})$ is given by

$$f_*\mu(C) := \mu(f^{-1}(C)).$$

Definition 1.6.9 (Almost Surely)**.** We say that a property P holds **μ-almost surely** if and only if there is a set $A \in \mathfrak{A}$ such that for all $a \in A$ we have $P(a)$ and $\mu(A^c) = 0$. We sometimes also say that **μ-almost all** points satisfy P.

Theorem 1.6.10 (Uniqueness Theorem)**.** *Suppose that μ is a probability and let ν be another probability on $(X, \mathfrak{A})$. Suppose that $\mathfrak{E} \subseteq \mathfrak{A}$ is an intersection stable generator of $\mathfrak{A}$. If $\mu|_{\mathfrak{E}} = \nu|_{\mathfrak{E}}$ then $\mu = \nu$.*[25]

Sketch of proof. Note that

$$\mathfrak{D} := \{A \in \mathfrak{A} \mid \nu(A) = \mu(A)\}$$

is a Dynkin system. As $\mathfrak{E}$ is a π-system the Dynkin-π-λ-Theorem 1.6.5 yields that $\mathfrak{D}$ is a σ-algebra. As $\mathfrak{E} \subseteq \mathfrak{D}$ we conclude $\mathfrak{D} = \mathfrak{A}$ □

Definition 1.6.11. By $\mathfrak{B}(\mathbb{R})$ we denote the measurable space $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the standard σ-algebra on $\mathbb{R}$.[26] This notation will be generalized in Definition 1.8.4.

Lemma 1.6.12 (Generalized Transformation Formula)**.** *Let $(X, \mathfrak{A})$ and $(Y, \mathfrak{F})$ be measurable spaces and $\varphi : (X, \mathfrak{A}) \to (Y, \mathfrak{F})$ measurable. Further, let $f : (Y, \mathfrak{F}) \to \mathfrak{B}(\mathbb{R})$ be measurable. Then f is $\varphi_*\mu$ integrable if and only if $f \circ \varphi$ is μ integrable. Further we have*

$$\int_A f \, \mathrm{d}\varphi_*\mu = \int_{\varphi^{-1}(A)} f \circ \varphi \, \mathrm{d}\mu$$

for any $A \in \mathfrak{A}$.

Sketch of proof. We prove the result for $f = \mathbb{1}_B$ for $B \in \mathfrak{A}$.

$$\begin{aligned} \int_A \mathbb{1}_B \, \mathrm{d}\varphi_*\mu &= \int \mathbb{1}_{A \cap B} \, \mathrm{d}\varphi_*\mu = \mu\left(\varphi^{-1}(A \cap B)\right) \\ &= \int \mathbb{1}_{\varphi^{-1}(A \cap B)} \, \mathrm{d}\mu = \int_{\varphi^{-1}(A)} \mathbb{1}_{\varphi^{-1}(B)} \, \mathrm{d}\mu \\ &= \int_{\varphi^{-1}(A)} \mathbb{1}_B \circ \varphi \, \mathrm{d}\mu . \end{aligned}$$

[25] This is Theorem II.5.6 in [Elstrodt, 2011, p.60].

[26] This is the Borel-σ-algebra as in Definition 1.8.1.

It follows for step functions by linearity. Then one can approximate any measurable function from below by step functions. □

Proposition 1.6.13 (Chebyshev's Inequality)**.** *Suppose that μ is a probability. Let $\varepsilon > 0$ and f be measurable and fix $0 < p < \infty$. Then for any $\varepsilon > 0$ we have*

$$\mu\left(\{x \in X \mid |f(x)| > \varepsilon\}\right) \leqslant \frac{1}{\varepsilon^p} \int |f|^p \, \mathrm{d}\mu\,.$$

Proof. The proof is a simple calculation:

$$\varepsilon^p \cdot \mu\left(\{x \in X \mid |f(x)| > \varepsilon\}\right) \leqslant \int_{\{|f|>\varepsilon\}} |f|^p \, \mathrm{d}\mu \leqslant \int |f|^p \, \mathrm{d}\mu\,.$$ □

1.6.3 Types of Convergence

Let $(X, \mathfrak{A})$ be a measurable space and μ a probability. Let $(I, <)$ be a directed set and $\mathbf{f} = (f_i)_{i\in I}$ be a net of measurable functions $X \to \mathbb{R}$. There are several notions of convergence for $\mathbf{f}$.

Definition 1.6.14 (Almost Sure Convergence)**.** $\mathbf{f}$ is said to **converge μ-almost surely** to f if and only if there is $A \in \mathfrak{A}$ with $\mu(A^c) = 0$ such that

$$\forall x \in A : f_i(x) \xrightarrow[i\in I]{} f(x)\,.$$

Definition 1.6.15 (Almost Surely Uniform Convergence)**.** Let $\mathbf{f} \in \mathcal{F}(X, \mathbb{R})^{\mathbb{N}}$. We call $\mathbf{f}$ almost surely uniformly convergent to $f \in \mathcal{F}(X, \mathbb{R})$ if and only if for any $\delta > 0$ there is $A \in \mathfrak{A}$ such that $\mu(A) < \delta$ and $\mathbf{f}|_{A^c}$ converges uniformly.

Theorem 1.6.16 (Egorov's Theorem)**.** *Assume that $\mathbf{f} \in \mathcal{M}(X, \mathbb{R})^{\mathbb{N}}$ converges μ-almost surely to $f \in \mathcal{M}(X, \mathbb{R})$. Then $\mathbf{f}$ converges μ-almost surely uniformly to f.*[27]

Definition 1.6.17 ($L^p(\mu)$)**.** For any measure ν on $(X, \mathfrak{A})$ and $p \geqslant 1$ we denote by

$$L^p(\nu) := \left\{ f \in \mathcal{M}(X, \mathbb{R}) \;\middle|\; \int |f|^p \, \mathrm{d}\nu < \infty \right\}$$

the vector space of all p-integrable functions. Equipped with the norm

$$\|f\|_p := \int |f|^p \, \mathrm{d}\nu$$

$L^p(\nu)$ becomes a Banach space.

Definition 1.6.18 (L^p-convergence)**.** Suppose that $f_i \in L^p(\mu)$ for all $i \in I$. We call $\mathbf{f}$ **L^p-convergent** to f if and only if f_i converges with respect to $\|\cdot\|_p$, i.e.

$$\int |f_i - f|^p \, \mathrm{d}\mu \xrightarrow[i\in I]{} 0\,.$$

[27] This is Theorem VI.3.5 in [Elstrodt, 2011, p.252].

Definition 1.6.19 (Convergence in Probability)**.** We call **f convergent in probability** to f if and only if

$$\forall \varepsilon > 0 : \lim_{i \in I} \mu\left(\{x \in X \mid |f_i(x) - f(x)| > \varepsilon\}\right) = 0\,.$$

It is easy to see (CHEBYSHEV's Inequality) that L^p convergence implies convergence in probability. If looking at sequences instead of general nets we further have that almost sure convergence implies convergence in probability. This can be seen by EGOROV'S THEOREM 1.6.16. The converse holds if one descends to a subsequence.

Lemma 1.6.20. *Any sequence convergent in probability has an almost surely convergent subsequence.*

Proof. Let $A^i_\delta := \{x \in X \mid |f_i(x) - f(x)| > \delta\}$. For any $\delta > 0$ there is $K_\delta \in \mathbb{N}$ such that for any $k > K_\delta$ we have $\mu\left(A^k_\delta\right) < \delta$. Clearly

$$\sum_{n=1}^{\infty} \mu\left(A^{K_{2^{-n}}}_{2^{-n}}\right) = 1 < \infty\,.$$

Letting $B_k = A^{K_{2^{-n}}}_{2^{-n}}$ we learn by BOREL-CANTELLI-LEMMA that

$$\mu\left(\limsup_{n \to \infty} B_k\right) = 0\,.$$

Thus μ-almost all points are only in finitely many B_k. This means that μ-almost all $x \in X$ are such that for large n we have

$$\left|f_{K_{2^{-n}}}(x) - f(x)\right| < 2^{-n}\,.$$

Thus $n \mapsto f_{K_{2^{-n}}}$ is almost surely convergent. □

An important question is under what circumstances one can interchange limits and integrals. A seminal result also called Dominated Convergence Theorem is

Theorem 1.6.21 (Lebesgue Convergence Theorem)**.** *Let ν be any measure on $(X, \mathfrak{A})$ Let $(f_n)_{n \in \mathbb{N}} \in (\mathcal{M}(X, \mathbb{R}))^{\mathbb{N}}$ be a sequence of measurable functions. Further let $f : X \to \mathbb{R}$ be measurable and suppose that the sequence $f_n(x)$ converges to $f(x)$ for ν-almost all $x \in X$. Further let $g : X \to \mathbb{R}^+_0$ be non-negative and integrable such that $|f_n| \leqslant g$. Then we have*

$$\lim_{n \to \infty} \int_X f_n \, \mathrm{d}\nu = \int_X \lim_{n \to \infty} f_n \, \mathrm{d}\nu = \int_X f \, \mathrm{d}\nu\,.$$

1.6.4 Conditional Expectation

Let $(X, \mathfrak{A}, \mu)$ be a probability space and $\mathfrak{F} \leq \mathfrak{A}$ be a sub-σ-algebra. Further let $T \in \mathcal{M}(X, \mathbb{R})$ be any observable.

Definition 1.6.22 (Conditional Expectation)**.** We call a measurable map $E : (X, \mathfrak{F}) \to \mathfrak{B}(\mathbb{R})$ a **conditional expectation** for T if and only if

$$\int_F E \, \mathrm{d}\mu = \int_F T \, \mathrm{d}\mu$$

for any $F \in \mathfrak{F}$.

Remark 1.6.22.1. A conditional expectation is the *"best fit"* of an observable to the *"information"* in the sub-σ-algebra.

Proposition 1.6.23. *The conditional expectation is μ-almost surely unique.*[28]

We will denote the conditional expectation by $\mathbb{E}_\mu [T \mid \mathfrak{F}]$.

Definition 1.6.24. We equip the vector space $L^2(\mu)$ with the scalar product

$$\langle f, g \rangle := \int f \cdot g \, \mathrm{d}\mu \, .$$

This turns $L^2(\mu)$ into a Hilbert space.

Lemma 1.6.25. *Let $f \in L^1(\mu)$ and suppose that $g \in L^\infty(\mu)$ is $\mathfrak{F}$-measurable. Then*

$$\mathbb{E}_\mu [f \cdot g \mid \mathfrak{F}] = g \cdot \mathbb{E} [f \mid \mathfrak{F}] \, .$$[29]

Sketch of proof. Let $A \in \mathfrak{F}$. Observe that

$$\int_F \mathbb{1}_A \cdot \mathbb{E}_\mu [f \mid \mathfrak{F}] \, \mathrm{d}\mu = \int_{F \cap A} \mathbb{E}_\mu [f \mid \mathfrak{F}] \, \mathrm{d}\mu = \int_{F \cap A} f \, \mathrm{d}\mu = \int_F \mathbb{1}_A \cdot f \, \mathrm{d}\mu$$

for $F \in \mathfrak{F}$. So the statement holds for indicator functions. By approximation we obtain the result in full generality. □

Proposition 1.6.26. *Let $V \leq L^2(\mu)$ be the subspace of functions measurable w.r.t. $\mathfrak{F}$. The map $\Psi : L^2(\mu) \to V$, $f \mapsto \mathbb{E} [f \mid \mathfrak{F}]$ is the orthogonal projection.*[30]

Lemma 1.6.27 (Tower Property)**.** *Let $\mathfrak{F}_2 \leq \mathfrak{F}_1 \leq \mathfrak{A}$ be a chain of sub-σ-algebras. Then for any $f \in L^1(\mu)$ we have*

$$\mathbb{E}_\mu [\mathbb{E}_\mu [f \mid \mathfrak{F}_1] \mathfrak{F}_2] = \mathbb{E}_\mu [f \mid \mathfrak{F}_2] = \mathbb{E}_\mu [\mathbb{E}_\mu [f \mid \mathfrak{F}_2] \mathfrak{F}_1] \, .$$[31]

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[28] This can for example be found as Theorem 5.1. in [Einsiedler and Ward, 2011, p.121].
[29] This is part (3) of Theorem 5.1. in [Einsiedler and Ward, 2011, p.121].
[30] This can implicitly found in the presentations on page 124 in [Einsiedler and Ward, 2011].
[31] This is part (4) of Theorem 5.1. in [Einsiedler and Ward, 2011, p.121].

1.7 Topology

Let X and Y be topological spaces. Recall that by $\mathcal{C}(X,Y)$ we denote the set of continuous functions from X to Y. The set of all continuous functions $X \to \mathbb{R}$ with compact support is denoted by $\mathcal{C}_c(X,\mathbb{R})$. If τ is the topology on X and $x \in X$ we denote by the neighbourhood filter of x by

$$\mathcal{U}(x) := \{V \in \mathfrak{P}(X) \mid \exists U \in \tau : x \in U \subseteq V\} .$$

Similarly we define for $A \subset X$ the system of neighbourhoods

$$\mathcal{U}(A) := \{V \in \mathfrak{P}(X) \mid \exists U \in \tau : A \subseteq U \subseteq V\}$$

The interior and closure of $A \subseteq X$ is denoted by $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ respectively.

Recall that for a metric space (X,d) we will denote by $B_\varepsilon(x)$ the open ball of radius ε around x, i.e.

$$B_\varepsilon(x) := \{x' \in X \mid d(x,x') < \varepsilon\} .$$

Definition 1.7.1 (Normal Space)**.** X is called **normal** if and only if for each pair $(A,B) \in \mathfrak{P}(X)$ of closed and disjoint subsets there are disjoint open sets $U_A \supseteq A$ and $U_B \supseteq B$.

Theorem 1.7.2 (Urysohn's Lemma)**.** *Let X be normal. For any pair of closed and disjoint subsets $(A,B) \in \mathfrak{P}(X)$ there is a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.*[32]

Lemma 1.7.3. *If X is compact* Hausdorff *then X is normal.*

Proof. Let $A, B \subseteq X$ be closed and disjoint. For each $a \in A$ and $b \in B$ there are disjoint open sets $U_{a,b} \ni a$ and $V_{a,b} \ni b$. Then $\{U_{a,b} \mid a \in A\}$ is an open cover of A. So by compactness there is $n \in \mathbb{N}$ and $\{a_1, \ldots, a_n\} \subseteq A$ such that $U_b := \bigcup_{i=1}^n U_{a_i,b} \supseteq A$. Then $V_b := \bigcap_{i=1}^n V_{a_i,b}$ is an open neighbourhood of b. So $\{V_b \mid b \in B\}$ is an open cover of B. Again by compactness there is $m \in \mathbb{N}$ and $\{b_1, \ldots, b_m\}$ such that $V := \bigcup_{i=1}^m V_b \supseteq B$. Note that $U := \bigcap_{i=1}^m U_{b_i} \supseteq A$ is disjoint from V. □

Definition 1.7.4 (Residuality)**.** We call a set $A \subseteq X$ residual, if and only if A can be written as an intersection of countably many subsets with dense interior.

Theorem 1.7.5 (Baire Category Theorem)**.** *If X is a locally compact* Hausdorff *space and A residual then A is dense.*[33]

1.8 Topology and Measure Theory

Let X and Y be topological spaces and τ the topology on X.

Definition 1.8.1 (Borel-σ-algebra)**.** The **Borel-σ-algebra** on X is the σ-algebra generated by open sets, i.e. $\sigma(\tau)$.

[32] This is Lemma 4 in chapter 4 of [Kelley, 1975, p.115].

[33] For example found in [Kelley, 1975, p.200].

Lemma 1.8.2. *If $f : X \to Y$ is continuous, then f is measurable with respect to the* Borel*-σ-algebras.*

Proof. This trivially follows from the Generator Theorem 1.6.2. □

This association of measurable spaces to topological spaces can be viewed as a functor:

Definition 1.8.3 (The category PreMeas)**.** Let PreMeas denote the category of all measurable spaces with morphisms given by the measurable maps.

Definition 1.8.4 (Borel Functor)**.** Let $\mathfrak{B}$ be the functor

$$\mathfrak{B} : \mathsf{Top} \longrightarrow \mathsf{PreMeas},\ (Z, \tau) \longmapsto (Z, \sigma(\tau))\,,$$

such that each morphism f in Top gets send to the unique morphism $\mathfrak{B}f$ in PreMeas such that $U_{\mathsf{Top}}(f) = U_{\mathsf{PreMeas}}(\mathfrak{B}f)$.[34] We will omit the $\mathfrak{B}$ most of the time and write f instead of $\mathfrak{B}f$.

Also we will usually omit the brackets $\mathfrak{B}X = \mathfrak{B}(X)$.

Definition 1.8.5 (Borel measure)**.** A measure which is defined on the Borel-σ-algebra is called a **Borel measure**.

1.8.1 Regularity of Measures

Here I again reuse parts of my bachelor's thesis [Haupt, 2020, p.20 et seq.] Let τ be the topology on X and μ a measure on $\mathfrak{B}X$.

Definition 1.8.6 (Local-Finiteness and Regularity)**.** a) The measure μ is **locally finite** if and only if

$$\forall x \in X : \exists V \in \mathcal{U}(x) : \mu(V) < \infty\,.$$

b) A set $A \in \mathfrak{A}$ is called **inner regular (w.r.t. μ)** if and only if

$$\mu(A) = \sup\{\mu(K) \mid K \text{ compact with } K \subseteq A\}\ .$$

We call μ inner regular if and only if all $A \in \mathfrak{A}$ are inner regular.

c) A set $A \in \mathfrak{A}$ is called **outer regular** if and only if

$$\mu(A) = \inf\{\mu(U) \mid U \in \tau \text{ with } U \supseteq A\}\ .$$

If and only if all $A \in \mathfrak{A}$ are outer regular we call μ outer regular.

d) A set $A \in \mathfrak{A}$ is called **regular**, if and only if A is inner and outer regular. An outer and inner regular measure μ is called regular.

[34] See Remark 1.2.4.1 for a discussion of the forgetful functors.

Proposition 1.8.7. *Let X be a compact* Hausdorff *space with* Borel-σ*-algebra* $\mathfrak{A}$. *Let μ be a regular measure on $(X, \mathfrak{A})$. Suppose that for all measurable $A \in \mathfrak{A}$ we have $\mu(A) \in \{0, 1\}$, then either $\mu = 0$ or there is $x \in X$ such that $\mu = \delta_x$.*

Proof. Suppose that there is no x such that $\mu = \delta_x$. Then $\mu(\{x\}) = 0$ for $x \in X$. By regularity there must be an open set U_x with $x \in U_x$ and $\mu(U_x) = 0$ for $x \in X$. Clearly $\{U_x \mid x \in X\}$ is an open cover. By compactness there is a finite subcover $\{U_{x_i} \mid i \in \{1, \ldots, n\}\}$. Calculate

$$\mu(X) = \mu\left(\bigcup_{i=1}^{n} U_{x_i}\right) \leqslant \sum_{i=1}^{n} \mu(U_{x_i}) = 0\,.$$

So $\mu = 0$. □

Definition 1.8.8 (Polish Space)**.** A topological space X is called **polish** if and only if X is separable and metrizable with a complete metric.

Theorem 1.8.9 (Theorem of Ulam)**.** *If X is a Polish space and μ locally finite, then μ is regular.*[35]

1.8.2 Borel Spaces

Definition 1.8.10 (Borel Space)**.** X is called **Borel** if and only if X is homeomorphic to a Borel measurable subset of a Polish space.

Definition 1.8.11. Let X be a Borel space. A subset $A \subset X$ is called **analytic** if and only if there is an uncountably infinite Borel space Y and a Borel subset $B \subset X \times Y$ such that $A = \text{proj}_X(B)$.

Remark 1.8.11.1. The graph of a continuous function $f \in \mathcal{C}(Y, X)$ is Borel measurable. Furthermore, $f(B) = \text{proj}_X(\text{Graph}(f) \cap B \times X)$. Thus, all the continuous images of Borel subsets of Polish spaces are analytic. In particular all Borel measurable sets in Borel spaces are analytic.[36]

Definition 1.8.12. A subset $A \subset X$ is called **analytically measurable** if and only if A is an element of the σ-algebra generated by the analytic sets. A function $f : X \to Y$ is called **analytically measurable** if and only if preimages of Borel-subsets are analytically measurable.

Theorem 1.8.13 (Jankov-von Neumann Selection Theorem)**.** *Let X and Y be* Borel *spaces and $A \subseteq X \times Y$ be analytic. There exists an analytically measurable function $\varphi : \text{proj}_X(A) \to Y$ such that* $\text{Graph}(\varphi) \subseteq A$.[37]

Definition 1.8.14 (Universally Measurable)**.** Let X be a Polish space and $\mathfrak{A}$ the Borel-σ-algebra. Recall that we denote the completion of $\mathfrak{A}$ w.r.t. a measure μ by $\mathfrak{A}_\mu$. A set $A \subseteq$ is called **universally measurable** if and only if $A \in \mathfrak{A}_\mu$ for all Borel probability measures μ.

Theorem 1.8.15. *Every analytically measurable set is universally measurable.*[38]

[35] Statement and proof can be found as Theorem VIII.1.16 in [Elstrodt, 2011, p.320 et seq.].

[36] See also [Bertsekas, 1996, Proposition 7.36].

[37] This is Proposition 7.49. in Chapter 7.7 of [Bertsekas, 1996]. In order to match the notions of *"analytical"* take a look at Chapter 7.6.1 in [Bertsekas, 1996].

[38] This is Corollary 7.42.1 in [Bertsekas, 1996, p. 169].

1.8.3 Riesz Representation Theorem

Let X locally compact HAUSDORFF and $\mathfrak{A}$ be the BOREL-σ-algebra on X. Recall that $\mathcal{C}_c(X, \mathbb{R})$ denotes the set of continuous functions with compact support.

Definition 1.8.16 (Positive Linear Form)**.** A linear form $I : \mathcal{C}_c(X, \mathbb{R}) \to \mathbb{R}$ is called **positive** if and only if for all non-negative f (i.e. $\forall x \in X : f(x) \geqslant 0$) we have $I(f) \geqslant 0$.

Theorem 1.8.17 (RIESZ Representation Theorem)**.** *If $I : \mathcal{C}_c(X, \mathbb{R}) \to \mathbb{R}$ is a positive linear form then there exists a unique inner regular* BOREL *measure μ such that*[39]

$$\forall f \in \mathcal{C}_c(X, \mathbb{R}) : \int f \, \mathrm{d}\mu = I(f) \, .$$

Corollary 1.8.17.1. *Any inner regular* BOREL *measure μ is uniquely determined by the positive linear form*

$$I_\mu : \mathcal{C}_c(X, \mathbb{R}) \longrightarrow \mathbb{R}, \; f \longmapsto \int f \, \mathrm{d}\mu \, .$$

1.8.4 Regular Conditional Probabilities

The seminal paper named *"Existence of Conditional Probabilities"* written by J. HOFFMANN JØRGENSEN [Hoffmann-Jørgensen, 1971] provides exactly what we need in this work.

Definition 1.8.18 (Regular Conditional Probability)**.** Let $(X, \mathfrak{A}, \mu)$ be a probability space and $(Y, \mathfrak{F})$ be a measurable space. For a given measurable map

$$\pi : (X, \mathfrak{A}) \to (Y, \mathfrak{F})$$

we let $\nu := \pi_* \mu$. We denote by $\mathfrak{F}_\nu$ the completion of $\mathfrak{F}$ w.r.t. ν. A map

$$R : \mathfrak{F} \times Y \longrightarrow [0, 1]$$

is called a **regular conditional probability** of μ with respect to π if and only if

(a) $A \mapsto R(A, y)$ is a regular probability measure on $(X, \mathfrak{A})$ for all $y \in Y$.

(b) $y \mapsto R(A, y)$ is $(Y, \mathfrak{F}_\mu) \to \mathfrak{B}[0, 1]$ measurable for any $A \in \mathfrak{A}$.

(c) For any $A \in \mathfrak{A}$ and $F \in \mathfrak{F}$ we have

$$\int_F R(A, y) \, \mathrm{d}\nu(y) = \mu \left(A \cap \pi^{-1}(F) \right) \, .$$

Theorem 1.8.19. *Suppose that X be a* HAUSDORFF *space and $\mathcal{B}$ its* BOREL-*σ-algebra, i.e. $(X, \mathcal{B}) = \mathfrak{B}X$. Let μ be a regular probability measure on $\mathfrak{B}X$. Assume that $(Y, \mathfrak{F})$ is a measurable space. For any measurable function $\pi : \mathfrak{B}X \to (Y, \mathfrak{F})$ there exists a regular conditional probability R.*[40]

[39] This well-known result can be found as Theorem VIII.2.5 in the book [Elstrodt, 2011, p.335].
[40] This is Theorem 1 in [Hoffmann-Jørgensen, 1971, p.260].

It is natural to require the measures $R(\cdot, y)$ to be supported on the fibers $\pi^{-1}(\{y\})$.

Theorem 1.8.20. *Let $(X, \mathfrak{A}, \mu)$ be any probability space. For some measurable space $(Y, \mathfrak{F})$ let there be a measurable map $\pi : (X, \mathfrak{A}) \to (Y, \mathfrak{F})$. Suppose that there exists a regular conditional probability measure R. Further suppose that the graph of π is measurable, i.e.* $\mathrm{Graph}(\pi) \in \mathfrak{A} \otimes \mathfrak{F}$. *Then we have for $\nu := \pi_* \mu$ that*

(a) $\pi(X) \in \mathfrak{F}_\mu$ *and* $\nu(\pi(X)) = 1$.

(b) $\forall y \in \pi(X) : \{y\} \in \mathfrak{F}$.

(c) For ν-almost all $y \in Y$ we have

$$R(\pi^{-1}(\{y\}), y) = 1 \, .$$

(d) There exists a regular conditional probability R_0 of μ given π such that[41]

$$\forall y \in \pi(X) : R_0(\pi^{-1}(\{y\}), y) = 1 \, .$$

1.8.5 Extending Measures

Let $(X, \mathfrak{A}, \mu)$ be a measure space.

Definition 1.8.21 (Extension)**.** Suppose $\mathfrak{F} \geq \mathfrak{A}$ is a super-σ-algebra, i.e. a σ-algebra which contains $\mathfrak{A}$. A measure $\tilde{\mu}$ on $\mathfrak{F}$ is called an **extension** of μ if and only if $\tilde{\mu}|_{\mathfrak{A}} = \mu$.

One very natural extension is the completion:

Definition 1.8.22 (Completion)**.** Let $\mathfrak{F}$ be the σ-algebra generated by $\mathfrak{A}$ and every subset of sets with measure zero, i.e.

$$\mathfrak{F} := \sigma\left(\mathfrak{A} \cup \{N \subseteq X \mid \exists A \in \mathfrak{A} : \mu(A) = 0 \text{ and } N \subseteq A\}\right) \, .$$

All the $F \in \mathfrak{F}$ have a corresponding $A \in \mathfrak{A}$ such that the symmetric difference $F \triangle A$ is contained in a set $C \in \mathfrak{A}$ of measure zero. The extended measure $\tilde{\mu}$ now simply says $\tilde{\mu}(F) = \mu(A)$. There is no other way to extend μ to a measure on $\mathfrak{F}$. The completion of $\mathfrak{A}$ w.r.t. μ shall be denoted by $\mathfrak{A}_\mu$.

If one extends measures to super-σ-algebras of the completion the uniqueness is lost. Not every chain of extensions has an upper bound as shown by

Example 1.8.22.1. Consider the natural numbers $\mathbb{N}$ equipped with σ-algebras

$$\mathfrak{A}_n := \mathfrak{P}(\{1, \ldots, n\}) \cup \{\mathbb{N}, \emptyset, \{n+1, n+2, \ldots, \infty\}\} \, .$$

On those σ-algebras define probability measures

$$\mu_n := \delta_{n+1}$$

which means that only the infinite set $\{n+1, \ldots, \infty\}$ gets weight. It is easy to see that those measures form a chain of extensions. However defining a *"measure"* on $\sigma\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right) = \mathfrak{P}(\mathbb{N})$, via $\mu(A) = \mu_n(A)$ if $A \in \mathfrak{A}_n$ gives a set function that is zero on all finite sets and thus not σ-additive.

[41] This is Theorem 2 in [Hoffmann-Jørgensen, 1971, p. 262].

One method to construct extensions beyond the completion is to choose some measure preserving map $\pi : (X, \mathfrak{A}, \mu) \to (Y, \Sigma, \nu)$ *"conditionalize"* and then *"complete fiber-wise"*: Suppose there is a regular conditional probability R w.r.t. π. Let $\mathfrak{A}_y$ be the completion of $\mathfrak{A}$ w.r.t. the measure $R(\cdot, y)$. Further let Σ_ν be the completion of Σ w.r.t. ν. Then you can define a system of subsets

Definition 1.8.23.

$$\begin{aligned}\mathfrak{A} \dagger R := \Big\{ A \subseteq X \;\Big|\; & \exists B \in \Sigma : \nu(B) = 1 \text{ and} \\ & A \in \bigcap_{y \in B} \mathfrak{A}_y \text{ and} \\ & R(A, \cdot) : B \to [0,1] \text{ is } \Sigma_\nu\text{-measurable.} \Big\} .\end{aligned} \tag{3}$$

Lemma 1.8.24. *$\mathfrak{A} \dagger R$ is a Dynkin system.*

Proof. **$X \in \mathfrak{A} \dagger R$:** Clearly $X \in \mathfrak{A}_y$ and $R(X, y) = 1$ for any $y \in Y$.

$A \in \mathfrak{A} \dagger R \Rightarrow A^c \in \mathfrak{A} \dagger R$: All the $\mathfrak{A}_y$ are σ-algebras. The intersection $\bigcap_{y \in B_A} \mathfrak{A}_y$ is also a σ-algebra. As $A \in \bigcap_{y \in B_A} \mathfrak{A}_y$ we have $A^c \in \bigcap_{y \in B_A} \mathfrak{A}_y$. Clearly $y \mapsto R(A^c, y) = 1 - R(A, y)$ is measurable.

Countable disjoint unions: Let $(A_n)_{n \in \mathbb{N}} \in (\mathfrak{A} \dagger R)^{\mathbb{N}}$ be a disjoint sequence. Note that $B = \bigcap_{n \in \mathbb{N}} B_{A_n}$ has full measure. Clearly for any $n \in \mathbb{N}$ we have that

$$A_n \in \bigcap_{y \in B} \mathfrak{A}_y .$$

As the latter is a σ-algebra, we conclude that the union is also contained in it. Now note that

$$R\left(\bigcup_{n \in \mathbb{N}} A_n, y \right) = \sum_{i=1}^{n} R(A_n, y) .$$

The pointwise limit of a sequence of measurable functions is again measurable. So we have $\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A} \dagger R$. □

Lemma 1.8.25. *For any regular conditional probability R we have*

$$\mathfrak{A}_\mu \subseteq \mathfrak{A} \dagger R .$$

Proof. Let $A \in \mathfrak{A}_\mu$. Then there is $A_1, A_2 \in \mathfrak{A}$ such that $A_1 \subseteq A \subseteq A_2$ and $\mu(A_1) = \mu(A_2)$. On the one hand, monotonicity yields

$$R(A_1, y) \leqslant R(A_2, y)$$

for any $y \in Y$. On the other hand

$$\int R(A_1, y) \, \mathrm{d}\nu(y) = \int R(A_2, y) \, \mathrm{d}\nu(y) .$$

Thus, $R(A_1, y) = R(A_2, y)$ for μ-almost all $y \in Y$. Furthermore, the pointwise defined function $R(A, \cdot)$ coincides μ-almost surely with the measurable function $R(A_1, \cdot)$. This shows that A is μ-almost surely contained in $\mathfrak{A}_y$ and $R(A, \cdot)$ is μ-almost surely measurable. So $A \in \mathfrak{A} \dagger R$. □

Definition 1.8.26. Let $\mathfrak{F}$ be any σ-algebra such that $\mathfrak{A}_\mu \subseteq \mathfrak{F} \subseteq \mathfrak{A} \dagger R$. Define an extension μ^* of μ onto $\mathfrak{F}$ by

$$\mu^*(F) := \int R(F, y)\, \mathrm{d}\nu(y)\,.$$

1.8.6 Measure Algebras

In [Walters, 2000, p.54] Peter Walters introduces the concept of *"measure algebras"*. Similar to the extensions discussed above this helps overcoming measurability issues.

Definition 1.8.27 (Boolean σ-algebra)**.** Let A be a set, $0_A, 1_A \in A$ be elements and $\hat{\neg} : A \to A$, $\hat{\wedge} : A \times A \to A$ and $\hat{\vee} : A \times A \to A$ be operations. We call the six-tuple $(A, \hat{\wedge}, \hat{\vee}, \hat{\neg}, 0_A, 1_A)$ is called a **Boolean algebra** if and only if

a) $\hat{\wedge}$ and $\hat{\vee}$ are associative.

b) $\hat{\wedge}$ and $\hat{\vee}$ are commutative.

c) $a\hat{\vee}(a\hat{\wedge}b) = a$ and $a\hat{\wedge}(a\hat{\vee}b) = a$ for $a, b \in A$.

d) $a\hat{\vee}0_A = a$ and $a\hat{\vee}1_A = a$ for $a \in A$.

e) $\hat{\wedge}$ and $\hat{\vee}$ are distributive in both ways.

f) $a\hat{\vee}\hat{\neg}a = 1_A$ and $a\hat{\wedge}\hat{\neg}a = 0_A$

We call the six-tuple $(A, \hat{\wedge}, \hat{\vee}, \hat{\neg}, 0_A, 1_A)$ a **Boolean σ-algebra** if and only if it is a Boolean algebra and $\hat{\vee}$ and $\hat{\wedge}$ are also defined for a countably infinite number of arguments.

If A_1 and A_2 are Boolean σ-algebras we call an bijection $\Psi : A_1 \to A_2$ an **isomorphism** if and only if $\Psi(\hat{\bigvee}_{n=1}^\infty a_n) = \hat{\bigvee}_{n=1}^\infty \Psi(a_n)$, $\Psi(\hat{\bigwedge}_{n=1}^\infty a_n) = \hat{\bigwedge}_{n=1}^\infty \Psi(a_n)$, $\Psi(\hat{\neg}a) = \hat{\neg}\Psi(a)$, $\Psi(1_{A_1}) = 1_{A_2}$ as well as $\Psi(0_{A_1}) = 0_{A_2}$.

Let $(X, \mathfrak{A}, \mu)$ be a probability space.

Definition 1.8.28 (Measure Algebra)**.** Define an equivalence relation of $\mathfrak{A}$ by

$$A \sim_\mu B :\Longleftrightarrow \mu(A \triangle B) = 0\,.$$

Let $[A]_\mu$ denote the equivalence class of A. The quotient $\mathsf{ma}(\mathfrak{A}, \mu) := \mathfrak{A}/\sim_\mu$ is called the **(associated) measure algebra**. We define a σ-additive map

$$\tilde{\mu} : \mathsf{ma}(\mathfrak{A}, \mu) \longrightarrow \mathbb{R}_0^+,\ \tilde{\mu}([A]_\mu) = \mu(A)\,.$$

Lemma 1.8.29. $\mathsf{ma}(\mathfrak{A}, \mu)$ *is a* Boolean *σ-algebra with*

$$\begin{aligned} [A]_\mu \hat{\vee} [B]_\mu &:= [A \cup B]_\mu \\ [A]_\mu \hat{\wedge} [B]_\mu &:= [A \cup B]_\mu \\ \hat{\neg}[B]_\mu &:= [B^c]_\mu \\ 0_A &:= [\emptyset]_\mu \\ 1_A &:= [X]_\mu \end{aligned}$$

Definition 1.8.30 (Conjugacy of Probability Spaces)**.** We call two probability spaces $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{F}, \nu)$ **conjugate** if and only if there is an isomorphism Ψ between $\mathsf{ma}(\mathfrak{A}, \mu)$ and $\mathsf{ma}(\mathfrak{F}, \nu)$ such that $\tilde{\nu}(\Psi(A)) = \tilde{\mu}(A)$ for $A \in \mathsf{ma}(\mathfrak{A}, \mu)$.

We can relate extensions and conjugacies in one special case:

Definition 1.8.31 (Strongly Approximable Extension)**.** Let $(X, \mathfrak{A}, \mu)$ be a probability space. Let $(X, \mathfrak{A}^*, \mu^*)$ be an extension. We call $A \in \mathfrak{A}^*$ **strongly approximable** if and only if there is $B \in \mathfrak{A}$ such that $\mu^*(A \triangle B) = 0$. We call the extension **strongly approximable** if and only if every $A \in \mathfrak{A}^*$ is strongly approximable.

Lemma 1.8.32. *If an extension $(X, \mathfrak{A}^*, \mu^*)$ is strongly approximable then*

$$\mathsf{ma}(\mathfrak{A}^*, \mu^*) \cong \mathsf{ma}(\mathfrak{A}, \mu) .$$

Proof. For $A \in \mathfrak{A}^*$ there is $B_A \in \mathfrak{A}$ such that $[A]_{\mu^*} = [B_A]_{\mu^*}$. Define

$$\Psi : \mathsf{ma}(\mathfrak{A}^*, \mu^*) \longrightarrow \mathsf{ma}(\mathfrak{A}, \mu), [A]_{\mu^*} \longmapsto [B_A]_{\mu} .$$

It is straightforward to see that Ψ is an isomorphism. □

Lemma 1.8.33. *Let $(X, \mathfrak{A}, \mu)$ be a probability space. Given $\mathfrak{E} \subseteq \mathfrak{P}(X)$ define $\mathfrak{A}^* := \sigma(\mathfrak{A} \cup \mathfrak{E})$. Suppose that there is an measure μ^* on $\mathfrak{A}^*$ extending μ such that any $E \in \mathfrak{E}$ is strongly approximable. Then the extension $(X, \mathfrak{A}^*, \mu^*)$ is strongly approximable.*

Proof. Let $\mathfrak{F}$ be the system of all strongly approximable $A \in \mathfrak{A}^*$. Clearly $\mathfrak{A} \subseteq \mathfrak{F}$ and $\mathfrak{E} \subseteq \mathfrak{F}$. We show that $\mathfrak{F}$ is a σ-algebra. Recall that $A \triangle B = A^c \triangle B^c$, so $\mathfrak{F}$ is closed under complements. Let $(A_n)_{n \in \mathbb{N}} \in \mathfrak{F}^{\mathbb{N}}$ be disjoint. For $n \in \mathbb{N}$ pick $B_n \in \mathfrak{A}$ such that $\nu(A_n \triangle B_n) = 0$. Now

$$\left(\bigcup_{n \in \mathbb{N}} A_n \right) \triangle \left(\bigcup_{n \in \mathbb{N}} B_n \right) \subseteq \bigcup_{n \in \mathbb{N}} (A_n \triangle B_n) .$$

So $\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{F}$. So $\mathfrak{F} \subseteq \mathfrak{A}^*$ is a σ-algebra containing $\mathfrak{A} \cup \mathfrak{E}$, so $\mathfrak{F} = \mathfrak{A}^*$. □

Proposition 1.8.34. *Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{F}, \nu)$ be probability spaces with extensions $(X, \mathfrak{A}^*, \mu^*)$ and $(Y, \mathfrak{F}^*, \nu^*)$. Let $(Y, \mathfrak{F}^*, \nu^*)$ be strongly approximable and $\Psi : \mathsf{ma}(\mathfrak{F}^*, \nu^*) \to \mathsf{ma}(\mathfrak{F}, \nu)$ an isomorphism. Any isomorphism $\Phi : (X, \mathfrak{A}^*, \mu^*) \to (Y, \mathfrak{F}^*, \nu^*)$ induces an isomorphism of* Boolean *algebras*

$$\tilde{\Phi} : \mathsf{ma}(\mathfrak{A}, \mu) \to \mathsf{ma}(\mathfrak{F}, \nu), \ \tilde{\Phi}([A]_{\mu}) = \Psi([\Phi(A)]_{\nu^*}) .$$

Proof. Firstly, we show that $\tilde{\Phi}$ is well-defined. Let $A \sim_{\mu} B$, i.e. $\mu(A \triangle B) = 0$. Note that $\Phi(A \triangle B) = \Phi(A) \triangle \Phi(B)$ and thus

$$\nu(\Phi(A) \triangle \Phi(B)) = \nu(\Phi(A \triangle B)) = \mu(A \triangle B) = 0$$

by measure preservation. So $\tilde{\Phi}$ is well-defined.

Lastly, we show that $\tilde{\Phi}$ is an isomorphism of Boolean σ-algebras. This however easily follows from the above and the fact that Φ is an almost surely bijective and bi-measurable map $(X, \mathfrak{A}, \mu) \to (Y, \mathfrak{A}, \nu)$. □

Corollary 1.8.34.1. *Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$. Let $\mathfrak{A}^* := \sigma(\mathfrak{A} \cup \{X_0\})$ and $\mathfrak{F}^* := \sigma(\mathfrak{F} \cup \{Y_0\})$. Suppose there are extensions μ^* and ν^* on to $\mathfrak{A}^*$ and $\mathfrak{F}^*$ such that $\mu^*(X_0) = 1 = \nu^*(Y_0)$. Any isomorphism Φ between $(X, \mathfrak{A}^*, \mu^*)$ and $(Y, \mathfrak{F}^*, \nu^*)$ induces an isomorphism of* Boolean *algebras between $\mathsf{ma}(X, \mathfrak{A}, \mu)$ and $\mathsf{ma}(\mathfrak{F}, \nu)$.*

1.8.7 Baire-σ-Algebra

Besides the **Borel**-σ-algebra there is another canonical σ-algebra on topological spaces.

Definition 1.8.35 (G_δ)**.** Let (X,τ) be a topological space. A subset $A \subseteq X$ is called $\mathbf{G}_\delta$ if and only if there exists a sequence $(O_n)_{n\in\mathbb{N}} \in \tau^{\mathbb{N}}$ such that $A = \bigcap_{n\in\mathbb{N}} O_n$.

Definition 1.8.36 (Baire-σ-algebra)**.** Let X be a compact Hausdorff space. The **Baire-σ-algebra** is the σ-algebra generated by the family of compact G_δ sets.

Remark 1.8.36.1.

1. There are non-equivalent definitions of *"Baire set"* in the literature. Especially in the more general setting on locally compact Hausdorff spaces the notions differ.
2. The Baire-σ-algebra is a sub-σ-algebra of the Borel-σ-algebra. This can be seen as countable intersections of open sets are Borel measurable. So any G_δ set is Borel measurable. In particular the Borel-σ-algebra contains the family of compact G_δ sets.
3. In compact metric spaces any closed subset C is G_δ. For example one can take the intersection over the $\frac{1}{n}$ balls around C. Thus in the setting of compact metric spaces both σ-algebras coincide.
4. Any compact Hausdorff space X is compact and G_δ. So the σ-ring[42] generated by the compact G_δ sets coincides with the Baire-σ-algebra. This remains true for locally compact σ-compact spaces.

Proposition 1.8.37. *Let X and Y be compact* Hausdorff *and $f : X \to Y$ be continuous. Then f is measurable with respect to the* Baire*-σ-algebras on X and Y.*

Proof. Observe that continuous preimages of G_δ sets are again G_δ. Now note that compact sets are closed in Hausdorff spaces. Further, continuous preimages of closed sets are closed. Finally, closed subsets of compact spaces are compact. We conclude that continuous preimages of compact sets are compact. $\square$

Following the procedure presented in [Elstrodt, 2011, p.17] we use Transfinite Induction 1.3.27 to find a very useful presentation of the Baire-σ-algebra via limits of continuous functions.

We start by defining families of functions via Transfinite Recursion 1.3.28:

Definition 1.8.38. Let $\mathcal{F}_0 := \mathcal{C}(X,[0,1])$. Suppose that for some ordinal $\mathfrak{a}$ the family $\mathcal{F}_\mathfrak{a}$ is already defined. Then we define

$$\mathcal{F}_{\mathfrak{a}+1} := \left\{ f \;\middle|\; \exists\, (f_n)_{n\in\mathbb{N}} \in \mathcal{F}_\mathfrak{a}^{\mathbb{N}} : f = \lim_{n\to\infty} f_n \text{ pointwise} \right\}.$$

For a limit ordinal $\mathfrak{b}$ suppose that all the families $\mathcal{F}_\mathfrak{a}$ are defined for $\mathfrak{a} < \mathfrak{b}$. Then we define

$$\mathcal{F}_\mathfrak{b} := \bigcup_{\mathfrak{a}<\mathfrak{b}} \mathcal{F}_\mathfrak{a}.$$

[42] See Definition I.3.6 in [Elstrodt, 2011, p.13].

Clearly for any ordinal $\mathfrak{b} > \mathfrak{a}$ we have $\mathcal{F}_\mathfrak{a} \subseteq \mathcal{F}_\mathfrak{b}$. It is an easy exercise to use Transfinite Induction 1.3.27 in order to prove the following lemmata:

Lemma 1.8.39. *For any ordinal $\mathfrak{a}$ the family $\mathcal{F}_\mathfrak{a}$ is closed under multiplication.*

Lemma 1.8.40. *For any ordinal $\mathfrak{a}$ and $f, g \in \mathcal{F}_\mathfrak{a}$ we have that if $f \leqslant g$ then $g - f \in \mathcal{F}_\mathfrak{a}$.*

Lemma 1.8.41. *For any ordinal $\mathfrak{a}$ and $f, g \in \mathcal{F}_\mathfrak{a}$ if $f + g \leqslant 1$ then $f + g \in \mathcal{F}_\mathfrak{a}$.*

For convenience of the reader the proof of Lemma 1.8.39 is presented. The omitted proofs are similar.

Proof of Lemma 1.8.39. The statement clearly holds for $\mathfrak{a} = 0$. Now suppose it holds for some $\mathfrak{a} > 0$. Pointwise convergence preserves multiplication. So the statement also holds for $\mathfrak{a} + 1$. Now suppose $\mathfrak{b}$ is a limit ordinal and the statement holds for any $\mathfrak{a} < \mathfrak{b}$. Then $f, g \in \mathcal{F}_\mathfrak{b}$ means that there is $\mathfrak{a}_f, \mathfrak{a}_g$ such that $f \in \mathcal{F}_{\mathfrak{a}_f}$ and $g \in \mathcal{F}_{\mathfrak{a}_g}$. Let $\mathfrak{a} := \max(\mathfrak{a}_f, \mathfrak{a}_g)$. Then $f, g \in \mathcal{F}_\mathfrak{a}$ and thus $f \cdot g \in \mathcal{F}_\mathfrak{a} \subseteq \mathcal{F}_\mathfrak{b}$. □

Now we extract from those families of functions families of sets.

Definition 1.8.42. For any ordinal $\mathfrak{a}$ we define

$$\mathcal{A}_\mathfrak{a} := \{A \subseteq X \mid \mathbb{1}_A \in \mathcal{F}_\mathfrak{a}\} .$$

Clearly if $\mathfrak{a} < \mathfrak{b}$ then $\mathcal{A}_\mathfrak{a} \subseteq \mathcal{A}_\mathfrak{b}$. The lemmata about the families of functions translate into

Lemma 1.8.43. *For any ordinal $\mathfrak{a}$ the family $\mathcal{A}_\mathfrak{a}$ is a π-system.*

Proof. Note that $\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$. Thus it follows from Lemma 1.8.39. □

Lemma 1.8.44. *For any ordinal $\mathfrak{a}$ and $A, B \in \mathcal{A}_\mathfrak{a}$ we have $A \setminus B \in \mathcal{A}_\mathfrak{a}$.*

Proof. $\mathcal{A}_\mathfrak{a}$ is closed under intersection. So w.l.o.g. we can assume $B \subseteq A$. Then $\mathbb{1}_B \leqslant \mathbb{1}_A$ and thus $\mathbb{1}_{A \setminus B} = \mathbb{1}_A - \mathbb{1}_B \in \mathcal{F}_\mathfrak{a}$. Thus, $A \setminus B \in \mathcal{A}_\mathfrak{a}$. □

We can use those results to prove

Proposition 1.8.45. *For any ordinal $\mathfrak{a}$ we have that*

$$\left\{ \biguplus_{n \in \mathbb{N}} B_n \;\middle|\; B_n = A_n \setminus A'_n \text{ for } A_n, A'_n \in \mathcal{A}_\mathfrak{a} \text{ and } B_n \text{ disjoint} \right\} \subseteq \mathcal{A}_{\mathfrak{a}+1} .$$

Proof. Let $(A_k)_{k \in \mathbb{N}} \in \mathcal{A}_\mathfrak{a}^\mathbb{N}$ be a sequence of disjoint sets. For any $i, j \in \mathbb{N}$ we have $\mathbb{1}_{A_i} + \mathbb{1}_{A_j} = \mathbb{1}_{A_i \cup A_j} \leqslant 1$. Furthermore,

$$\sum_{i=1}^{n} \mathbb{1}_{A_i} \in \mathcal{F}_\mathfrak{a}$$

for any $n \in \mathbb{N}$. Observe that

$$\mathbb{1}_{\biguplus_{n\in\mathbb{N}} A_n} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{1}_{A_i} .$$

In total we conclude that

$$\left\{ \biguplus_{n\in\mathbb{N}} A_n \;\middle|\; A_n \in \mathcal{A}_{\mathfrak{a}} \right\} \subseteq \mathcal{A}_{\mathfrak{a}+1} .$$

As $\mathcal{A}_{\mathfrak{a}}$ is preserved by taking differences of sets this concludes the proof. □

With those Lemmata and Propositions at hand we can prove

Theorem 1.8.46. *Let Ω be the smallest uncountable ordinal. Then $\mathcal{A}_\Omega$ is a Dynkin-system.*

Proof. As Ω is a limit-ordinal we have that

$$\mathcal{A}_\Omega = \bigcup_{\mathfrak{a}<\Omega} \mathcal{A}_{\mathfrak{a}} .$$

We will prove the properties of a Dynkin-system one by one.

$X \in \mathcal{A}_\Omega$: Clearly $\mathbb{1}_X$ is continuous and thus $\mathbb{1}_X \in \mathcal{F}_0 \subseteq \mathcal{F}_\Omega$.

Closure under complements: Let $A \in \mathcal{A}_\Omega$. There is $\mathfrak{a} < \Omega$ such that $A \in \mathcal{A}_{\mathfrak{a}}$. By the above $A^c = X \setminus A \in \mathcal{A}_{\mathfrak{a}} \subseteq \mathcal{A}_\Omega$.

Closure under countable disjoint union: Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}_\Omega^{\mathbb{N}}$ be a disjoint sequence of sets. Then for $n \in \mathbb{N}$ there is $\mathfrak{a}_n < \Omega$ such that $A_n \in \mathcal{A}_{\mathfrak{a}_n}$. Define

$$\mathfrak{b} := \min \{ \mathfrak{o} \leqslant \Omega \mid \forall n \in \mathbb{N} : \mathfrak{a}_n \leqslant \mathfrak{o} \}$$

This $\mathfrak{b}$ exists as the ordinals are well-ordered and as the set is non-empty as it contains Ω. Ω is the first uncountable ordinal so all the ordinals $\mathfrak{a}_n$ are countable. The countable union of countable sets is countable. This implies that

$$\mathfrak{r} := \bigcup_{n\in\mathbb{N}} \mathfrak{a}_n$$

is countable. By Lemma 1.3.22 $\mathfrak{r}$ is an ordinal. Clearly we see that $\mathfrak{r} > \mathfrak{a}_n$ for all $n \in \mathbb{N}$. So $\mathfrak{b} \leqslant \mathfrak{r} < \Omega$. In fact we even have $\mathfrak{b} = \mathfrak{r}$. Obviously, $A_n \in \mathcal{A}_{\mathfrak{b}}$ for any $n \in \mathbb{N}$ and $\mathfrak{b} + 1 < \Omega$. So Proposition 1.8.45 yields

$$\biguplus_{n\in\mathbb{N}} A_n \in \mathcal{A}_{\mathfrak{b}+1} \subseteq \mathcal{A}_\Omega .$$

□

Remark 1.8.46.1. We use the fact that Ω is the smallest uncountable ordinal only for the closure under countable disjoint union. There we use the insight that Ω can not be reached by a sequence of smaller ordinals. With the same idea we can see that $\mathcal{F}_\Omega$ is closed under limits of sequences.

Corollary 1.8.46.1. *$\mathcal{A}_\Omega$ is the* BAIRE-*σ-algebra on X.*

Proof. Let $\mathfrak{K}$ denote the system of compact G_δ sets. $\mathfrak{K}$ is a π-system. We now show that $\mathfrak{K} \subseteq \mathcal{A}_\Omega$. Let $A \in \mathfrak{K}$. There is a sequence of open sets V_n such that $A = \bigcap_{n\in\mathbb{N}} V_n$. Let $U_n := \bigcap_{i=1}^n V_n$. Then $U_n \supseteq U_{n+1}$. By the URYSOHN's Lemma 1.7.2 there are continuous functions $f_n : X \to [0,1]$ such that $f_n|_A = 1$ and $f_n|_{U_n^c} = 0$. The sequence f_n converges pointwise to $\mathbb{1}_A$ as $n \to \infty$. This means that $A \in \mathcal{A}_1$. Furthermore, $\mathfrak{K} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_\Omega$. The DYNKIN-$\pi$-$\lambda$-THEOREM 1.6.5 implies that the λ-system generated by $\mathfrak{K}$ is a σ-algebra and thus the BAIRE-σ-algebra. However as $\mathfrak{A}_\Omega$ is itself a λ-system containing $\mathfrak{K}$ the λ-system generated by $\mathfrak{K}$ is contained in $\mathfrak{A}_\Omega$. This means that the BAIRE-σ-algebra is contained in $\mathcal{A}_\Omega$.

Conversely, the continuous functions are BAIRE measurable. The system of measurable functions (with respect to any σ-algebra) is closed under pointwise limits of sequences. So a trivial Transfinite Induction 1.3.27 shows that for any ordinal $\mathfrak{a}$ all the functions in $\mathcal{F}_\mathfrak{a}$ are BAIRE measurable. In particular any $f \in \mathcal{F}_\Omega$ is BAIRE measurable. This shows that any $A \in \mathcal{A}_\Omega$ is BAIRE measurable. □

Remark 1.8.46.2. a) Again recall the fact, that the system of measurable functions $\mathcal{M}(X, \mathbb{R})$ with respect to any σ-algebra is closed under pointwise convergence. Thus any σ-algebra $\mathfrak{A}$ that makes the continuous functions measurable must make all the functions in $\mathcal{F}_\Omega$ measurable. So it must hold that $\mathcal{A}_\Omega \subseteq \mathfrak{A}$. For compact HAUSDORFF spaces $\mathcal{A}_\Omega$ is a σ-algebra itself as the Theorem shows. So for compact HAUSDORFF spaces the BAIRE-σ-algebra is the smallest σ-algebra that makes all the continuous functions measurable.

b) $\mathcal{F}_\Omega$ is exactly the system $\mathcal{M}(X, [0,1])$ of BAIRE measurable functions from X to $[0,1]$. Clearly $\mathcal{F}_\Omega \subseteq \mathcal{M}(X, [0,1])$. Conversely, see that any finite linear combination of step functions, whose range is contained in $[0,1]$, is contained in $\mathcal{F}_\Omega$. Similarly to the proof of the closedness under countable unions we can see that $\mathcal{F}_\Omega$ is closed under pointwise convergence of sequences. So $\mathcal{F}_\Omega$ contains any pointwise limit of sequences of step-functions. However, step-functions are (even uniformly) dense in $\mathcal{M}(X, [0,1])$.[43]

c) We further see that as $\mathcal{F}_\Omega$ is closed under limits, for any $\mathfrak{b} > \Omega$ we have $\mathcal{F}_\mathfrak{b} = \mathcal{F}_\Omega$ and consequently $\mathcal{A}_\mathfrak{b} = \mathcal{A}_\Omega$.

Corollary 1.8.46.2. *Let $\mathfrak{h} \subseteq \mathcal{F}(X, [0,1])$ be a system of functions closed under pointwise convergence. When $\mathcal{C}(X, [0,1]) \subseteq \mathfrak{h}$ then $\mathfrak{h}$ contains all* BAIRE *measurable functions $X \to [0,1]$.*

Proof. By Corollary 1.8.46.1 the family $\mathcal{F}_\Omega$ contains all BAIRE measurable functions $X \to [0,1]$. By assumption $\mathcal{F}_0 \subseteq \mathfrak{h}$. Now if for any ordinal $\mathfrak{a}$ we have $\mathcal{F}_\mathfrak{a} \subseteq \mathfrak{h}$ then $\mathcal{F}_{\mathfrak{a}+1} \subseteq \mathfrak{h}$ as $\mathfrak{h}$ is closed by pointwise convergence. Let $\mathfrak{b}$ be a limit ordinal and $\mathcal{F}_\mathfrak{a} \subseteq \mathfrak{h}$ for any $\mathfrak{a} < \mathfrak{b}$. Then $\mathcal{F}_\mathfrak{b} = \bigcup_{\mathfrak{a}:\mathfrak{a}<\mathfrak{b}} \mathcal{F}_\mathfrak{a} \subseteq \mathfrak{h}$. So by Transfinite Induction 1.3.27 $\mathcal{F}_\Omega \subseteq \mathfrak{h}$. □

[43] See for example Theorem III.4.13 [Elstrodt, 2011, p.108].

1.9 Weak Topologies

Let k be either $\mathbb{R}$ or $\mathbb{C}$. Let (V, τ) be a topological k-vector space, i.e. a vector space equipped with a topology such that vector addition and scalar multiplication are continuous.

Definition 1.9.1 (Dual Space)**.** Equip the set

$$V' := \{\varphi : V \longrightarrow k \mid \varphi \text{ linear and continuous } \}$$

with pointwise vector addition and scalar multiplication. Then V' is again a k-vector space and we call it the (topological) dual space of (V, τ).

This dual space allows us to *"weaken"* the norm induced topology on V:

Definition 1.9.2 (Weak Topology)**.** Let τ_w be the coarsest topology on V such that all $\varphi \in V'$ are continuous. Then (V, τ_w) is a topological vector space and we call τ_w the **weak topology**.

Usually we want V' not only to be a vector space but to be a topological vector space.

Definition 1.9.3 (Operator Norm)**.** Let $(V, \|\cdot\|)$ be a normed vector space. Then

$$\|\varphi\|_{V'} := \sup_{v \in V} \frac{|\varphi(v)|}{\|v\|}$$

is a norm on V'. We call $\|\cdot\|_{V'}$ the **operator norm**. The topology generated by the operator is called **strong**.

Remark 1.9.3.1. In this setting the norm induces a topology τ' on V' making (V', τ') a topological vector space. Thus there is also a weak topology on V'.

On V' we can define yet a coarser topology:

Definition 1.9.4 (Weak-$*$-Topology)**.** Let $(V, \|\cdot\|)$ be any normed k-vector space. Equip V' with the strong topology τ and define V'' as the dual space of V'. For $v \in V$ let $\iota(v) : V' \to k, \varphi \mapsto \varphi(x)$ be the evaluation functional. Clearly $\iota : V \to V''$ Let τ_{w*} be the coarsest topology such that $\iota(v)$ is continuous for any $v \in V$. We call τ_{w*} the weak-$*$-topology on V'.

Remark 1.9.4.1. On V' we have now three topologies: The strong, the weak and the weak-$*$-topology. In the infinite dimensional case they are often not the same.

Let $(V, \|\cdot\|)$ be a normed vector space. Define the unit ball of V' to be $B := \{\varphi \in V' \mid \|\varphi\|_{V'} \leqslant 1\}$.

Theorem 1.9.5 (Banach-Alaoglu)**.** *B is compact w.r.t. the weak-$*$-topology.*[44]

[44] This result is well-known and can be found e.g. in [Einsiedler and Ward, 2017, p.256, Theorem 8.10].

Idea of the proof. The main idea is to embed V' into k^V and observe that the weak-$*$-topology on V' corresponds to the trace topology of the product topology on k^V. By $K^k_\varepsilon(0) := \{r \in k \mid |a| \leqslant 1\}$ denote the ε-ball around $0 \in k$. By TYCHONOFF's Theorem

$$K := \prod_{v \in V} K^k_{\|v\|}(0)$$

is compact. Observe that $B \subseteq K$ and conclude that B is compact by showing that B is closed. □

Theorem 1.9.6. *If V is separable then the subspace topology on $B \subseteq V'$ w.r.t. the weak-$*$-topology is metrizable.*[45]

1.10 Topological Groups

Definition 1.10.1 (Topological Group)**.** A group $(G, \cdot)$ equipped with a topology τ is called a **topological group** if and only if the multiplication

$$\cdot : G \times G \longrightarrow G,\ (g, h) \longmapsto g \cdot h$$

and inversion

$$^{-1} : G \longrightarrow G,\ g \longmapsto g^{-1}$$

are continuous w.r.t. τ.

Definition 1.10.2 (The category TopGrp)**.** Let Grp denote the category of groups with group homomorphisms as morphisms. Further, let TopGrp denote the category of topological groups with continuous group homomorphisms as morphisms.

Definition 1.10.3 (Opposite Group)**.** Let $(G, \cdot)$ be a group. Define

$$* : G \times G \longrightarrow G,\ a * b := b \cdot a\,.$$

Clearly $(G, *)$ is a group. We call $(G, \cdot)^{\mathsf{op}} := (G, *)$ the **opposite group**. Whenever we notationally suppress the group operation $\cdot$ we also write G^{op} in order to denote the opposite group. Given a group homomorphism $f : G \to H$ we define $f^{\mathsf{op}} := f$ and obtain a group homomorphism $f : G^{\mathsf{op}} \to H^{\mathsf{op}}$. We obtain a functor

$$\mathsf{op} : \mathsf{Grp} \longrightarrow \mathsf{Grp}\,.$$

When G carries a topology then the same topology is a group topology on G^{op}. So we obtain a functor

$$\mathsf{op} : \mathsf{TopGrp} \longrightarrow \mathsf{TopGrp}\,.$$

Remark 1.10.3.1. The identity functor $\mathrm{Id}_{\mathsf{Grp}} : \mathsf{Grp} \to \mathsf{Grp}$ is naturally isomorphic to op.[46]

[45] This is Proposition 8.11. in [Einsiedler and Ward, 2017, p.257].

[46] This is an example in the Wikipedia-Page about Natural Transformations `https://en.wikipedia.org/wiki/Natural_transformation#Opposite_group`

Proof. We need to specify an isomorphism $\alpha(G)$ from G to G^{op} for every $G \in \mathsf{Grp}$ in order to construct the natural transformation $\alpha : \mathrm{Id}_{\mathsf{Grp}} \to \mathsf{op}$. Let $\alpha(G)$ be the map that sends $g \mapsto g^{-1}$. This clearly can be understood as an isomorphism $G \to G^{\mathsf{op}}$. Clearly $\alpha(G^{\mathsf{op}}) \circ \alpha(G) = \mathrm{Id}_G$ and $\alpha(G) \circ \alpha(G^{\mathsf{op}}) = \mathrm{Id}_{G^{\mathsf{op}}}$. So α is *"self inverse"*. It remains to show the naturality. Let $G, H \in \mathsf{Grp}$ and $f : G \to H$. Then

$$\alpha(H) \circ f(g) = f(g)^{-1} = f(g^{-1}) = f^{\mathsf{op}}(g^{-1}) = f^{\mathsf{op}} \circ \alpha(G)(g) .$$

Compare this to Figure 1. □

Definition 1.10.4 (Minkowski Product). Let G be a group. For $A, B \subset G$ we define the **Minkowski product**

$$A \cdot B = \{ab \mid a \in A, b \in B\} .$$

Definition 1.10.5 (Syndecity). Let G be topological group. A subset $A \subseteq G$ is called **left syndetic** if and only if there is a compact subset K such that $G = A \cdot K$. It is called **right syndetic** if and only if there is a compact subset K such that $G = K \cdot A$.

1.10.1 Haar Measure

Let G be a topological group.

Definition 1.10.6 (Locally Compact Group). We call G **locally compact**, if G is locally compact as a topological space, i.e.

$$\forall g \in G : \exists K \in \mathcal{U}(g) : K \textbf{ is compact.}$$

Definition 1.10.7 (Hausdorff Group). We call G **Hausdorff**, if G is Hausdorff as a topological space.

Definition 1.10.8 (Haar Measure). A measure m on $\mathfrak{B}G$ is called a **left Haar measure** if and only if it is left-invariant, i.e.

$$\forall A \in \sigma(\tau) : \forall g \in G : m(A) = m(gA) ,$$

and inner regular Borel. Similarly **right Haar measures** are right-invariant inner regular Borel-measures.

Theorem 1.10.9 (Existence and Uniqueness of a Haar Measure). *If G is locally compact, then there is a left Haar measure m. For any other left Haar measure m' there is $c \in \mathbb{R}^+$ such that $m' = c \cdot m$.*[47]

Definition 1.10.10 (Opposite Haar Measure). Define a measure m^{op} on $\mathfrak{B}G$ via $m^{\mathsf{op}}(A) := m(A^{-1})$. We call m^{op} the **opposite Haar measure**.

Lemma 1.10.11. *Let m be a left Haar measure. Then*

$$m^{\mathsf{op}}(A) := m(A^{-1})$$

is a right Haar measure. Further m^{op} is a left Haar measure on the opposite group G^{op}.

[47] This is Theorem VIII.3.12 in [Elstrodt, 2011, p.362].

Proof. Let $g \in G$ and $A \subseteq G$ be Borel measurable. Then

$$m^{\mathsf{op}}(Ag) = m\left((Ag)^{-1}\right) = m(g^{-1}A^{-1}) = m(A^{-1}) = m^{\mathsf{op}}(A)\,. \qquad \square$$

Definition 1.10.12 (Homogeneous Space)**.** Let $H \leq G$ be a closed subgroup. We call G/H a **homogeneous space**.

Remark 1.10.12.1. This naming convention reflects the fact that G acts on G/H transitively by left-multiplication. It further hints at the fact that if G is σ-compact then all algebraically transitive group actions induce a homogeneous space.[48]

Theorem 1.10.13. *Let G be a locally compact group and $H \leq G$ a closed subgroup. G acts on G/H by $\alpha(g, fH) := gfH$. If there is an α-invariant inner regular* Borel *probability measure μ on the homogeneous space G/H it is unique.*[49]

1.10.2 Pontryagin Duality

Let G be a locally compact Abelian group. We follow [Hewitt and Ross, 1963, Chapter Six].

Definition 1.10.14 (Character)**.** A **character** is a bounded continuous homomorphism $\chi : G \to \mathbb{C}^\times$. Here $\mathbb{C}^\times$ denotes the multiplicative group $(\mathbb{C} \setminus \{0\}\,, \cdot)$.

Remark 1.10.14.1. Let χ be a character. Suppose $|\chi(g)| \neq 1$ for some $g \in G$. By potentially exchanging g by g^{-1} we can assume that $|\chi(g)| > 1$. Then $|\chi(g^n)| = |\chi(g)|^n \xrightarrow{n\to\infty} \infty$. This is a contradiction to the boundedness. Thus $\chi : G \to \mathbb{S}^1$. Here $\mathbb{S}^1$ denotes the unit circle in the complex plane.

Definition 1.10.15 (The Pontryagin Dual Group)**.** The set of all characters of G equipped with pointwise multiplication is an Abelian group G^* and is called the **dual group**. We equip the dual group with the topology of uniform convergence on compact subsets and call it the **topological dual group**. We denote the dual group and the topological dual group by the same symbol G^*.

Proposition 1.10.16. *The topological dual group is locally compact Abelian.*[50]

Many properties of a topological group have their counterpart in the topological dual group, for example

Proposition 1.10.17. *If G is compact then G^* is discrete. If G is discrete then G^* is compact.*[51]

Theorem 1.10.18 (Pontryagin Duality)**.** *The canonical map into the bi-dual*

$$\Psi : G \longrightarrow (G^*)^*\,,\ g \longmapsto \Big[\chi \mapsto \chi(g)\Big]\,,$$

sending each element to the corresponding evaluation map, is an isomorphism of topological groups.[52]

[48] Cf. Proposition 2.44. in [Folland, 1995, p.55].
[49] This is a direct consequence of Theorem 2.49. in [Folland, 1995, p.57].
[50] This is Theorem 23.15 in [Hewitt and Ross, 1963, p.361].
[51] This is Theorem 23.17 in [Hewitt and Ross, 1963, p.362].
[52] This is Theorem 24.08 in [Hewitt and Ross, 1963, p.378].

1.10.3 Averaging Sequences

Suppose that G is locally compact with left Haar measure m.

We will work with sequences but there is an important reason to introduce the following notions for nets. In the proof of the later stated Corollary 1.13.2.1 of Theorem 1.13.2 we go over to a converging subnet. Note that in non-metrizable compact Hausdorff spaces it can happen that a sequence does not have a converging sub-**sequence** (but has a converging sub-net due to compactness). If X is non-metrizable then clearly the space of all probability measures on X is non-metrizable as well. So in this proof we will need to use nets instead of sequence even if we would try to only work with sequences.

Let $(I, \prec)$ be a directed set and $\mathcal{F} := (F_i)_{i\in I} \in \mathfrak{P}(G)^I$ a net of compact subsets.

Definition 1.10.19 (Ergodic Net)**.** We call $\mathcal{F}$ **left ergodic** if and only if for any $g \in G$ there is $K \in I$ such that for all $j \succ K$ we have $m(F_j) > 0$ and

$$\frac{m\left(F_j \triangle g \cdot F_j\right)}{m\left(F_j\right)} < \delta\,. \tag{4}$$

If in (4) we multiply g from the right instead of from the left and replace the left Haar measure m be a right Haar measure n we call $\mathcal{G}$ a **right ergodic net**.

If $(I, \prec) = (\mathbb{N}, <)$, we call $\mathcal{F}$ an **ergodic sequence**.

Definition 1.10.20 (Amenability)**.** We call G **left** (resp. **right**) **amenable** if and only if there is at least one left (resp. right) ergodic sequence $\mathcal{F} \in \mathfrak{P}(G)^{\mathbb{N}}$.

Definition 1.10.21 (Følner Net)**.** We call $\mathcal{F}$ **(left) Følner** if and only if for any compact set $K \subseteq G$ there is $K \in I$ such that for all $j \succ K$ we have $m(F_j) > 0$ and

$$\frac{m\left(F_j \triangle K \cdot F_j\right)}{m\left(F_j\right)} < \delta\,. \tag{5}$$

If in (5) we multiply K from the right instead of from the left and replace the left Haar measure m be a right Haar measure n we call $\mathcal{F}$ a **right Følner net**.

If $(I, \prec) = (\mathbb{N}, <)$, we call $\mathcal{F}$ a **Følner sequence**.

1.10.4 Densities

Still suppose that G is locally compact. Fix a left Haar measure m and a left ergodic net $\mathcal{F} = (F_i)_{i\in I}$. Let $\mathfrak{A}$ denote the Borel-σ-algebra of G.

Definition 1.10.22 ($\mathcal{F}$-Density)**.** We define the **upper $\mathcal{F}$-density** of a set $A \in \mathfrak{A}$ as

$$\bar{D}_{\mathcal{F}}(A) := \limsup_{i\in I} \frac{m\left(A \cap F_i\right)}{m(F_i)}\,.$$

The **lower $\mathcal{F}$-density** of A is then defined as

$$\underline{D}^{\mathcal{F}}(A) := \liminf_{i\in I} \frac{m\left(A \cap F_i\right)}{m(F_i)}\,.$$

If $\bar{D}_{\mathcal{F}}(A) = \underline{D}^{\mathcal{F}}(A)$ we define the **$\mathcal{F}$-density** of A as

$$D_{\mathcal{F}}(A) := \lim_{i \in I} \frac{m\left(A \cap F_i\right)}{m(F_i)}\,.$$

Remark 1.10.22.1. We will similarly define upper and lower densities for the right HAAR measure m^{op} and the right ergodic net $\mathcal{F}^{\mathsf{op}}$.

Upper- and lower densities are closely connected by the following result:

Lemma 1.10.23. *We have for any $A \in \mathfrak{A}$ that*

$$\bar{D}_{\mathcal{F}}(A) = 1 - \underline{D}^{\mathcal{F}}(A^c)\,.$$

Proof. We calculate

$$\begin{aligned}
\bar{D}_{\mathcal{F}}(A) &= \limsup_{i \in I} \frac{m\left(A \cap F_i\right)}{m(F_i)} \\
&= \limsup_{i \in I} \frac{m(F_i) - m\left(A^c \cap F_i\right)}{m(F_i)} \\
&= \limsup_{i \in I} \left(1 - \frac{m\left(A^c \cap F_i\right)}{m(F_i)}\right) \\
&= 1 - \liminf_{i \in I} \frac{m\left(A^c \cap F_i\right)}{m(F_i)} \\
&= 1 - \underline{D}^{\mathcal{F}}(A^c)\,.
\end{aligned}$$

□

Lemma 1.10.24. *Let m be a left HAAR measure and $\mathcal{G}$ be any left ergodic net. The upper $\mathcal{G}$-density w.r.t. m is invariant under left multiplication, i.e. for all $g \in G$ and $A \in \mathfrak{A}$ we have that*

$$\bar{D}_{\mathcal{G}}(g \cdot A) = \bar{D}_{\mathcal{G}}(A)\,.$$

Similarly, the upper $\mathcal{G}^{\mathsf{op}}$-density w.r.t. m^{op} is right invariant. The same holds for the lower density.

Proof. We calculate

$$\begin{aligned}
\bar{D}_{\mathcal{G}}(g \cdot A) - \bar{D}_{\mathcal{G}}(A) &= \limsup_{i \in I} \frac{m\left(g \cdot A \cap F_i\right)}{m(F_i)} - \limsup_{i \in I} \frac{m\left(A \cap F_i\right)}{m(F_i)} \\
&= \limsup_{i \in I} \frac{m\left(g^{-1} \cdot g \cdot A \cap g^{-1} \cdot F_i\right)}{m(F_i)} - \limsup_{i \in I} \frac{m\left(A \cap F_i\right)}{m(F_i)} \\
&= \limsup_{i \in I} \frac{m\left(g^{-1} \cdot g \cdot A \cap g^{-1} \cdot F_i\right)}{m(F_i)} + \liminf_{i \in I} -\frac{m\left(A \cap F_i\right)}{m(F_i)} \\
&\leqslant \limsup_{i \in I} \frac{m\left(A \cap g^{-1} \cdot F_i\right) - m\left(A \cap F_i\right)}{m(F_i)} \\
&\leqslant \limsup_{i \in I} \frac{m\left(A \cap \left(g^{-1} \cdot F_i \triangle F_i\right)\right)}{m(F_i)} \\
&\leqslant \limsup_{i \in I} \frac{m\left(g^{-1} \cdot F_i \triangle F_i\right)}{m(F_i)} \\
&= 0\,.
\end{aligned}$$

So $\bar{D}_{\mathcal{G}}(A) \geqslant \bar{D}_{\mathcal{G}}(g \cdot A)$. So for $B = g \cdot A$ we have

$$\bar{D}_{\mathcal{G}}(A) = \bar{D}_{\mathcal{G}}(g^{-1} \cdot B) \leqslant \bar{D}_{\mathcal{G}}(B) = \bar{D}_{\mathcal{G}}(g \cdot A)\,.$$

The statement about the lower density follows from Lemma 1.10.23. □

Lemma 1.10.25. *The upper density $\bar{D}_{\mathcal{F}}$ is subadditive as a map on the semigroup $(\mathfrak{P}(G), \cup)$, i.e. for any two sets $A, B \subseteq G$ we have*

$$\bar{D}_{\mathcal{F}}(A \cup B) \leqslant \bar{D}_{\mathcal{F}}(A) + D_{\mathcal{F}}(B)\,.$$

Proof. Let $A, B \in \mathfrak{A}$. We calculate

$$\begin{aligned}
\bar{D}_{\mathcal{F}}(A \cup B) &= \limsup_{i \in I} \frac{m\left((A \cup B) \cap F_i\right)}{m(F_i)} \\
&= \limsup_{i \in I} \frac{m\left((A \cap F_i) \cup (B \cap F_i)\right)}{m(F_i)} \\
&\leqslant \limsup_{i \in I} \frac{m\left(A \cap F_i\right) + m\left(B \cap F_i\right)}{m(F_i)} \\
&\leqslant \limsup_{i \in I} \frac{m\left(A \cap F_i\right)}{m(F_i)} + \limsup_{i \in I} \frac{m\left(B \cap F_i\right)}{m(F_i)} \\
&= \bar{D}_{\mathcal{F}}(A) + \bar{D}_{\mathcal{F}}(B)\,.
\end{aligned}$$

□

Definition 1.10.26 ($\mathcal{F}$-Banach Density)**.** For any $H \in \mathfrak{A}$ we define the **upper $\mathcal{F}$-Banach density** as

$$BD^*_{\mathcal{F}}(H) := \limsup_{i \in I} \sup_{h \in G} \frac{m(H \cap h \cdot F_i)}{m(F_i)}\,.$$

The **lower $\mathcal{F}$-Banach density** is defined as

$$BD^{\mathcal{F}}_*(H) := \liminf_{i \in I} \inf_{h \in G} \frac{m(H \cap h \cdot F_i)}{m(F_i)}\,.$$

If the lower and upper $\mathcal{F}$-Banach density coincide we simply call it **$\mathcal{F}$-Banach density** and denote it $BD_{\mathcal{F}}(H)$.

For the right Haar measure m^{op} and the right ergodic net $\mathcal{F}^{\mathsf{op}}$ we define

$$BD^*_{\mathcal{F}^{\mathsf{op}}}(H) := \limsup_{i \in I} \sup_{h \in G} \frac{m^{\mathsf{op}}(H \cap F_i^{-1} \cdot h)}{m^{\mathsf{op}}(F_i^{-1})}\,.$$

Note that here we multiply h from the right.

Lemma 1.10.27. *We have for any $H \in \mathfrak{A}$ that*

$$BD^*_{\mathcal{F}}(H) = 1 - BD^{\mathcal{F}}_*(H^c)\,.$$

Proof. We calculate

$$\begin{aligned}
BD^*_{\mathcal{F}}(H) &= \limsup_{i\in I} \sup_{h\in G} \frac{m\left(H \cap h\cdot F_i\right)}{m(F_i)} \\
&= \limsup_{i\in I} \sup_{h\in G} \frac{m(h\cdot F_i) - m\left(H^c \cap h\cdot F_i\right)}{m(F_i)} \\
&= \limsup_{i\in I} \sup_{h\in G} \left(1 - \frac{m\left(H^c \cap h\cdot F_i\right)}{m(F_i)}\right) \\
&= \mathbb{1} - \liminf_{i\in I} \inf_{h\in G} \frac{m\left(H^c \cap h\cdot F_i\right)}{m(F_i)} \\
&= 1 - BD^{\mathcal{F}}_*(H^c)\,.
\end{aligned}$$

□

1.11 Topological Dynamics

1.11.1 Basic Notions

Let G be a topological group.

Definition 1.11.1 (Continuous Group Action)**.** Let X be a topological space. An action α of G on X is called **(jointly) continuous** if and only if α is continuous as a map $G \times X \to X$.

Definition 1.11.2 (`tds`)**.** Let X be a topological space. Suppose that α is a continuous group action of G on X. We call the triple (X, G, α) a topological dynamical system (`tds`).

Let $\mathbf{X} = (X, G, \alpha)$ and $\mathbf{Y} = (Y, G, \beta)$ be two `tds`.

Definition 1.11.3 (Equivariance, Factor Maps)**.** A map $\pi : X \to Y$ is called **equivariant** if and only if

$$\forall g \in G : \forall x \in X : \pi(\alpha(g, x)) = \beta(g, \pi(x))\,.$$

An equivariant map π is called a **factor map** if and only if π is surjective. Factor maps are denoted as $\pi : \mathbf{X} \to \mathbf{Y}$.

Definition 1.11.4 (The Categories of Topological Dynamical Systems)**.**
Let $\mathsf{TopDyn}(G)$ denote the category of topological dynamical systems with the morphisms given by factor maps. Let CMet be the category of compact metric spaces with continuous maps as morphisms. By $\mathsf{CMetDyn}(G) \subset \mathsf{TopDyn}(G)$ we denote the full subcategory topological dynamical systems with compact metric phase space X. By $\mathsf{CHausDyn}(G) \subseteq \mathsf{TopDyn}(G)$ we denote the full subcategory of topological dynamical systems with compact Hausdorff phase space X.

Definition 1.11.5 (Opposite `tds`)**.** We define an action α^{op} of G^{op} on X by

$$\alpha^{\mathsf{op}}(g, x) = \alpha(g^{-1}, y)\,.$$

We call α^{op} the **opposite action** of α. Furthermore, we define the **opposite `tds`** $\mathbf{X}^{\mathsf{op}} := (X, G^{\mathsf{op}}, \alpha^{\mathsf{op}})$. Let $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. Define the factor map $\pi^{\mathsf{op}} : \mathbf{X}^{\mathsf{op}} \to \mathbf{Y}^{\mathsf{op}}$ by $\pi^{\mathsf{op}}(x) = \pi(x)$. The map π^{op} is indeed a factor map as

$$\pi^{\mathsf{op}}\left(\alpha^{\mathsf{op}}(g, x)\right) = \pi\left(\alpha(g^{-1}, x)\right) = \beta\left(g^{-1}, \pi(x)\right) = \beta^{\mathsf{op}}\left(g, \pi^{\mathsf{op}}(x)\right) .$$

We call π^{op} the **opposite factor map** of π. We obtain a functor

$$\mathsf{op} : \mathsf{TopDyn}(G) \longrightarrow \mathsf{TopDyn}(G^{\mathsf{op}}) .$$

1.11.2 Equicontinuity

Let $(X, G, \alpha), (Y, G, \beta) \in \mathsf{CHausDyn}(G)$. Denote $\mathbf{X} := (X, G, \alpha)$ and $\mathbf{Y} := (Y, G, \beta)$. Recall the Definition 1.5.1 of an equicontinuous family of functions.

Definition 1.11.6 (Equicontinuous System)**.** We call the system **X equicontinuous** if and only if the family

$$\alpha(G, \cdot) := \{\alpha(g, \cdot) : X \to X,\, x \mapsto \alpha(g, x) \mid g \in G\}$$

is equicontinuous.

Remark 1.11.6.1. If X is metrizable then equicontinuity is equivalent to to existence of an **invariant metric** d.

Definition 1.11.7 (The category $\mathsf{EquiDyn}(G)$)**.** Let $\mathsf{EquiDyn}(G) \subset \mathsf{CHausDyn}(G)$ denote the full subcategory of equicontinuous dynamical systems.

1.11.3 Minimality

Definition 1.11.8 (Minimal Subset)**.** A subset $A \subseteq X$ is called **(α)-minimal** if and only if A is invariant, compact and contains no non-empty closed and invariant subsets.

Remark 1.11.8.1. We will mostly suppress the action α notationally and simply call A minimal.

Lemma 1.11.9. *Every* $\mathbf{X} \in \mathsf{CHausDyn}(G)$ *has an α-minimal subset.*

Proof. Note that the intersection of invariant subsets is still invariant. Let $\mathfrak{H}$ denote the family of closed and invariant non-empty subsets of X. Let $\mathfrak{K} \subseteq \mathfrak{H}$ such that $(\mathfrak{K}, \subset)$ is linearly ordered. Then $K := \bigcap \mathfrak{K}$ is invariant and closed. By the Cantor Intersection Theorem K is non-empty. So any chain of closed and invariant non-empty subsets has a lower bound. By the Lemma of Zorn there is a minimal subset A of $(\mathfrak{H}, \subset)$. A is clearly α-minimal. □

Definition 1.11.10. We call the system **X minimal** if and only if X is α-minimal.

Proposition 1.11.11. *Let* **X** *be minimal. For any non-empty and invariant set* $A \subseteq X$ *we have* $\mathrm{cl}(A) = X$.

Proof. Let $\emptyset \neq A \subseteq X$ be invariant. Then $\mathrm{cl}(A)$ is invariant and closed. So $\mathrm{cl}(A) = X$ as X has no other non-empty invariant and closed subsets. $\square$

Corollary 1.11.11.1. *In minimal systems all orbits are dense.*

Definition 1.11.12 ((Topological) Transitivity)**.** We call **X (topologically) transitive** if there is at least one dense orbit.[53]

Remark 1.11.12.1. Note the difference between topologically transitive and algebraically transitive group actions.

1.11.4 Group Compactifications

Let G be a topological group. We follow [Hermle and Kreidler, 2022, p.7] in their presentation and definitions.

Definition 1.11.13 (Group Compactification)**.** Let K be a compact topological group and $\psi : G \to K$ be a continuous group homomorphism such that $\psi(G)$ is a dense subset of K. We then call the pair (K, ψ) a **group compactification**.

Remark 1.11.13.1. Note that we don't even require ψ to be injective, let alone an homeomorphism onto its range. This is relevant in our setting as we do not require the map $g \mapsto \alpha(g, \cdot) \in \mathcal{C}(X, X)$ to be injective.

Definition 1.11.14 (The Category of Group compactifications)**.** The category of group compactification $\mathsf{Comp}(G)$ consists of all group compactifications of G as objects. A morphism between two group compactifications (K_1, ψ_1) and (K_2, ψ_2) is a continuous group homomorphism $\Psi : K_1 \to K_2$ such that $\Psi \circ \psi_1 = \psi_2$.[54]

In order to connect dynamics with group compactifications the following example is paradigmatic.

Example 1.11.14.1. We consider the topological group $\mathbb{Z}$, where the topology is discrete. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the 1-dimensional torus. For any $\alpha \in [0, 1) \setminus \mathbb{Q}$ there is a group compactification $(\mathbb{T}, \psi_\alpha)$, where

$$\psi_\alpha : \mathbb{Z} \longrightarrow \mathbb{T},\ n \longmapsto (n \cdot \alpha) + \mathbb{Z} \in \mathbb{T} = \mathbb{R}/\mathbb{Z}\,.$$

We thus see how the irrational rotations give rise to group compactifications.

The way back from group compactifications to equicontinuous dynamics can be understood as a functor following Construction 4.11. in [Hermle and Kreidler, 2022, p.22]. For that we need to consider dynamical systems with a distinguished point.

Definition 1.11.15 (Pointed System)**.** Let (Z, G, γ) be a `tds` and $z \in Z$. We call the tuple (Z, G, γ, z) a **pointed** `tds`. A factor map between two pointed `tds` (Z, G, γ, z) and (Y, G, β, y) is a factor map π between the `tds` (Z, G, γ) and (Y, G, β) such that $\pi(z) = y$. The categories of pointed systems will be denoted by the subscript `pt`.

[53] Note that this definition is in some cases non-equivalent to the definition found in other parts of the literature.

[54] Cf. [Hermle and Kreidler, 2022, p.20]

Definition 1.11.16. Define a functor

$$\mathsf{Rot} : \mathsf{Comp}(G) \longrightarrow \mathsf{EquiDyn}_{\mathsf{pt}}(G),\ \mathsf{Rot}(K, \psi) := (K, G, \alpha_\psi, 1_K)$$

where

$$\alpha_\psi : G \times K \longrightarrow K,\ (g, k) \longmapsto \psi(g) \cdot k\,.$$

A morphism $H : \mathbf{K_1} \to \mathbf{K_2}$ between group compactifications is send to the unique factor map

$$\mathsf{Rot}(H) : \mathsf{Rot}(\mathbf{K_1}) \to \mathsf{Rot}(\mathsf{K}_2)$$

which satisfies $U_{\mathsf{Comp}(G)}(H) = U_{\mathsf{EquiDyn}(G)}(\mathsf{Rot}(H))$.[55]

Remark 1.11.16.1. Suppose that G is a subgroup of the group compactification K. Then $\Psi = \mathrm{Id}_G$ and $\mathsf{Rot}(K, \mathrm{Id}_G)$ is just the rotation on K by G.

1.12 Ergodic Theory

1.12.1 Basic Notions

Let G be a topological group.

Definition 1.12.1 (Invariant Probability)**.** Let $\alpha : G \times X \to X$ be a group action. A probability μ on $(X, \mathfrak{A})$ is called **α-invariant** if and only if for any $g \in G$ we have $\alpha(g, \cdot)_*\mu = \mu$. If the operation α is clear from the context we omit it notationally and call μ invariant.

Definition 1.12.2 (Ergodic Probability)**.** An invariant probability μ is called **ergodic** if and only if we have $\mu(A) \in \{0, 1\}$ for any α-invariant set $A \in \mathfrak{A}$.

Definition 1.12.3 (`mpds`)**.** The tuple $\mathbf{X} = (X, G, \alpha, \mu)$ is called a **measure preserving dynamical system** (`mpds`) if and only if $\alpha : G \times X \to X$ is a $\mathfrak{B}G \times \mathfrak{B}X \to \mathfrak{B}X$-measurable group action and μ an α-invariant probability. We call the system $\mathbf{X}$ (resp. the action α) **ergodic** if and only if μ is an ergodic probability.

Let $\mathbf{X} = (X, G, \alpha, \mu)$ and $\mathbf{Y} = (Y, G, \alpha, \nu)$ be two `mpds`.

Definition 1.12.4 ((Measurable) Factor Map)**.** An μ-almost surely defined measurable map $\pi : X \to Y$ is called **(measurable) factor map** if and only if π is equivariant and $\pi_*\mu = \nu$. If π is a factor map we will denote $\pi : \mathbf{X} \to \mathbf{Y}$.

Definition 1.12.5 (The Category $\mathsf{ProbDyn}(G)$)**.** Let $\mathsf{ProbDyn}(G)$ denote the category of measure preserving dynamical systems on probability spaces with measurable factor maps as morphisms.

[55] See Remark 1.2.4.1 for a discussion of the forgetful functors.

1.12.2 Mean Ergodic Theorem

Theorem 1.12.6 (Mean Ergodic Theorem)**.** *Define the sub-σ-algebra of α-invariant sets*

$$\mathfrak{F} := \{A \in \mathfrak{A} \mid \forall g \in G : \alpha(g, A) = A\} \ .$$

Let $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ be an ergodic sequence and m a HAAR *measure of G. Finally let μ be any α-invariant probability on X. For any $T \in \mathcal{M}(X, \mathbb{R})$ we have*

$$\frac{1}{m(F_n)} \int_{F_n} T \circ \alpha(g, \cdot) \, \mathrm{d}m(g) \xrightarrow[L^2(\mu)]{n\to\infty} \mathbb{E}_\mu [T \mid \mathfrak{F}] \ .$$

If μ is ergodic, then $\mathfrak{F}$ contains only sets of full or null measure, thus

$$\mathbb{E}_\mu [T \mid \mathfrak{F}] = \int T \, \mathrm{d}\mu$$

μ-almost surely.[56]

Remark 1.12.6.1. The space $L^2(\mu)$ is dense in $L^1(\mu)$. By HÖLDER's Inequality one can derive a version of the Mean Ergodic Theorem where L^2 is replaced by L^1.

1.12.3 Ergodic Decomposition

The book named *"Lectures on Choquet's Theorem"* written by ROBERT R. PHELPS [Phelps, 2001] utilises the eponymous CHOQUET'S THEOREM to provide proof to a general theorem about ergodic decomposition.

Ergodic Decomposition refers to the quest to somehow reduce the theory of invariant probabilities to ergodic probabilities by writing invariant probabilities as a convex combination (i.e. integral w.r.t. a probability).

Let Z be a compact HAUSDORFF space. Let $\mathfrak{F}$ be the **Baire**-σ-algebra on Z. For a family of continuous transformations $\mathcal{T} \subseteq \mathcal{C}(Z, Z)$ let $M_\mathcal{T}$ be the set of all probability measures μ on $\mathfrak{F}$ such that $T_*\mu = \mu$ for $T \in \mathcal{T}$. Equip $M_\mathcal{T}$ with the topology of weak-$*$-convergence.

We generalize the notion of ergodicity.

Definition 1.12.7. Let $A \subseteq Z$. We call A **$\mathcal{T}$-invariant** if and only if $A = T^{-1}(A)$ for any $T \in \mathcal{T}$.

Definition 1.12.8. We call $\mu \in M_\mathcal{T}$ **$\mathcal{T}$-ergodic** if and only if $\mu(A) \in \{0, 1\}$ for any $\mathcal{T}$-invariant BOREL measurable set A.

Theorem 1.12.9. *For any $\mu \in M_\mathcal{T}$ there exists a probability measure m_μ on the* ***Baire****-σ-algebra of $M_\mathcal{T}$ such that*

i) for any $f \in \mathcal{C}(Z, \mathbb{R})$ we have

$$\int_X f \, \mathrm{d}\mu = \int_{M_\mathcal{T}} \int_X f \, \mathrm{d}\eta \, \mathrm{d}m_\mu(\eta) \tag{6}$$

[56] If $\mathcal{F}$ is assumed to be Følner this is Theorem 8.13. in [Einsiedler and Ward, 2011, p.255]. It is however easy to see, that the proof given there generalizes to $\mathcal{F}$ being ergodic.

ii) and $m_\mu(B) = 0$ for any Baire*-subset B not containing any $\mathcal{T}$-ergodic probability.*[57]

Remark 1.12.9.1. a) The set of $\mathcal{T}$-ergodic probabilities is generally not a Baire subset of $M_\mathcal{T}$. Thus it is not clear whether it supports the probability.

Lemma 1.12.10. *Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}(Z,[0,1])^\mathbb{N}$ be a sequence of $\mathfrak{F}$-measurable functions converging pointwise to a measurable function f. Define for any $\mathfrak{F}$-measurable function $h : Z \to [0,1]$ the functional*

$$\hat{h} : M_\mathcal{T} \longrightarrow [0,1],\ \eta \longmapsto \int_X h \,\mathrm{d}\eta\,.$$

The sequence $\left(\hat{f}_n\right)_{n\in\mathbb{N}}$ converges pointwise to $\hat{f}$. Furthermore,

$$\begin{aligned}\int_{M_\mathcal{T}}\int_X f \,\mathrm{d}\eta\,\mathrm{d}m(\eta) &= \int_{M_\mathcal{T}}\int_X \lim_{n\to\infty} f_n \,\mathrm{d}\eta\,\mathrm{d}m(\eta)\\ &= \int_{M_\mathcal{T}} \lim_{n\to\infty}\int_X f_n \,\mathrm{d}\eta\,\mathrm{d}m(\mu)\\ &= \lim_{n\to\infty}\int_{M_\mathcal{T}}\int_X f_n \,\mathrm{d}\eta\,\mathrm{d}m(\eta)\end{aligned}$$

for any probability m on the Baire*-σ-algebra of $M_\mathcal{T}$.*

Proof. Note that $f_n \leqslant \mathbb{1}_Z$. $\mathbb{1}_Z$ is integrable with respect to any $\eta \in M_\mathcal{T}$. The Lebesgue Convergence Theorem 1.6.21 implies that $\hat{f}_n$ converges pointwise to $\hat{f}$. Further $\int f_n \,\mathrm{d}\eta \leqslant 1$ for any $\eta \in M_\mathcal{T}$. So we can again apply the Lebesgue Convergence Theorem 1.6.21 in order to finish the proof. □

Corollary 1.12.10.1. *Equation (6) holds for any $\mathfrak{F}$-measurable function.*

Proof. Let $\mathfrak{h}$ be the family of all functions that satisfy (6). By Lemma 1.12.10 $\mathfrak{h}$ is closed under pointwise convergence. By Theorem 1.12.9 $\mathcal{C}(Z,[0,1]) \subseteq \mathfrak{h}$. So Corollary 1.8.46.2 finishes the proof. □

Proposition 1.12.11. *Let M be any set of probabilities on Z equipped with the weak-$*$-topology. For any* Baire *measurable function $f : Z \to [0,1]$ the functional*

$$I_f : M \longrightarrow \mathbb{R},\ \eta \longmapsto \int f \,\mathrm{d}\eta$$

is measurable with respect to the Baire*-σ-algebra on M.*

Proof. Let $\mathfrak{h}$ be the set of all Baire measurable functions f such that I_f is Baire measurable. By the definition of the weak-$*$-topology

$$I_f : M \longrightarrow \mathbb{R}, \eta \longmapsto \int f \,\mathrm{d}\eta$$

[57] This is a Theorem in [Phelps, 2001, p.77].

is continuous for any $f \in \mathcal{C}(Z, [0,1])$. Continuous functions to the real numbers are Baire measurable. So $\mathcal{C}(Z, \mathbb{R}) \subseteq \mathfrak{h}$. By the Lemma 1.12.10 I_{g_n} converges pointwise to I_g whenever $g_n : Z \to [0,1]$ converges pointwise to g. Pointwise convergence preserves measurability so $\mathfrak{h}$ is closed under pointwise convergence. By Corollary 1.8.46.2 we learn that $\mathfrak{h}$ contains all Baire measurable functions. □

The following proposition is very helpful at deriving results from the Theorem 1.12.9.

Proposition 1.12.12. *Let $A \in \mathfrak{F}$. Then*

$$\{\eta \in M_{\mathcal{T}} \mid \eta(A) > 0\}$$

is a Baire *subset of $M_{\mathcal{T}}$.*

Proof. Clearly $\mathbb{1}_A$ is $\mathfrak{F}$-measurable. Hence by Proposition 1.12.11 $I_{\mathbb{1}_A}$ is Baire measurable. Thus,

$$\{\eta \in M_{\mathcal{T}} \mid \eta(A) > 0\} = I_{\mathbb{1}_A}^{-1}(\mathbb{R}^+)$$

is a Baire subset of $M_{\mathcal{T}}$. □

Remark 1.12.12.1. Clearly the proof shows that for any measurable set $B \subseteq \mathbb{R}$ we have that $\{\eta \in M_{\mathcal{T}} \mid \eta(A) \in B\}$ is Baire.

Corollary 1.12.12.1. *Let $A \in \mathfrak{F}$. If there is $\mu \in M_{\mathcal{T}}$ such that $\mu(A) > 0$, then there is a $\mathcal{T}$-ergodic probability ν such that $\nu(A) > 0$.*

Proof. We know that

$$0 < \mu(A) = \int_X \mathbb{1}_A \, \mathrm{d}\mu = \int_{M_{\mathcal{T}}} \int_X \mathbb{1}_A \, \mathrm{d}\eta \, \mathrm{d}m(\eta) = \int_{M_{\mathcal{T}}} \eta(A) \, \mathrm{d}m(\eta) \, .$$

By Proposition 1.12.12 we know that $P := \{\eta \in M_{\mathcal{T}} \mid \eta(A) > 0\}$ is Baire measurable. So we must have $m(P) > 0$. Theorem 1.12.9 yields that P must contain a $\mathcal{T}$-ergodic probability ν. □

Now consider again the `tds` $\mathbf{X} = (X, G, \alpha)$ with compact Hausdorff phase space X. We look at the direct self product $\mathbf{X} \otimes \mathbf{X}$. By Δ denote the diagonal in $X \times X$.

Proposition 1.12.13. *Let $A \subset X \times X$ be such that* $\mathrm{cl}(A) \cap \Delta = \emptyset$. *Then there is a compact* Baire *subset $B \supseteq A$ such $B \cap \Delta = \emptyset$.*

Proof. As $X \times X$ is normal we can use the Urysohn's Lemma 1.7.2. Let $f : X \times X \to [0,1]$ be continuous such that $f|_{\Delta} = 1$ and $f|_{\mathrm{cl}(A)} = 0$. Then as continuous functions are Baire measurable, we know that

$$B := f^{-1}\left(\left[0, \frac{1}{2}\right]\right)$$

is Baire. Clearly B is closed and disjoint from the diagonal. □

Corollary 1.12.13.1. *Let $A \subset X \times X$ be* Baire *measurable such that* $\mathrm{cl}(A) \cap \Delta = \emptyset$. *Suppose that for some invariant probability Θ on $\mathfrak{B}(X \times X)$ we have $\Theta(A) > 0$. Then there is an ergodic probability $\tilde{\mu}$ and a compact* Baire *measurable $B \supseteq A$ such that $B \cap \Delta = \emptyset$ and $\tilde{\mu}(B) > 0$.*

Proof. This is just a combination of Proposition 1.12.13 and Corollary 1.12.12.1. □

Corollary 1.12.13.2. *In Corollary 1.12.13.1 we can choose B to be compact and* Baire.

Let us see how regularity of measures works together with measures on the Baire-σ-algebra. A combination of [Dudley, 2002, Theorem 7.1.5] and [Dudley, 2002, Theorem 7.3.1] shows that

Theorem 1.12.14. *For any finite measure μ on a compact* Hausdorff *space being defined on a super-σ-algebra of the* Baire-σ-*algebra all the* Baire *sets are regular. Further, we can find a unique regular measure $\hat{\mu}$ defined on all* Borel *sets which coincides on the* Baire-σ-*algebra with μ.*

Remark 1.12.14.1. On compact sets $K \subseteq X$ this measure is given by $\hat{\mu}(K) = \inf \{\mu(A) \mid A \text{ is Baire and } K \subseteq A\}$.

1.12.4 Lindenstrauss Ergodic Theorem

Let G be locally compact, m a left Haar measure and $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ a left Følner sequence. Let α be an action of G by measure preserving transformations on a probability space $(X, \mathfrak{A}, \mu)$.

Definition 1.12.15 (Tempered Følner Sequence)**.** $\mathcal{F}$ is called **tempered** if and only if there is $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\frac{m\left(\bigcup_{k=1}^{n-1} F_k \cdot F_n\right)}{m\left(F_n\right)} < C\,. \tag{7}$$

Theorem 1.12.16 (Lindenstrauss Ergodic Theorem)**.** *Let $\mathcal{F}$ be tempered. For any $T \in L^1(\mu)$ there is an α-invariant function $\bar{T} \in L^1(\mu)$ such that for μ-almost all $x \in X$ we have*

$$\frac{1}{m(F_n)} \int_{F_n} T(\alpha(g,x))\, \mathrm{d}m(g) \xrightarrow{n \to \infty} \bar{T}(x)\,.\text{[58]} \tag{8}$$

1.13 Interplay of Topological Dynamics and Ergodic Theory

1.13.1 Existence of Invariant probabilities

Let G be left amenable and $(I, <)$ a directed set. Let m be a left Haar measure and $\mathcal{F} = (F_i)_{i \in I}$ a left ergodic net. Let $\mathbf{X} = (X, G, \alpha)$ be a `tds` and X be locally compact.

[58] This is Theorem 1.2 in [Lindenstrauss, 2001].

Proposition 1.13.1. *An inner regular* BOREL *probability μ on X is α-invariant if and only if*

$$\int f \, \mathrm{d}\mu = \int f \circ \alpha(g, \cdot) \, \mathrm{d}\mu \tag{9}$$

for any $f \in \mathcal{C}_c(X, \mathbb{R})$ and $g \in G$.

Proof. By the RIESZ Representation Theorem 1.8.17 we have that μ is invariant if and only if for all $g \in G$ we have

$$\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\alpha(g, \cdot)_* \mu \, .$$

Now by the Generalized Transformation Formula 1.6.12 this is equivalent to (9). □

Theorem 1.13.2 (Krylov-Bogolyubov Procedure)**.** *For a net $(\mu_i)_{i \in I}$ of probability measures define[59] ν_i by the formula*

$$\int f \, \mathrm{d}\nu_i := \frac{1}{m(F_i)} \int_{F_i} \int f \circ \alpha(g, \cdot) \, \mathrm{d}\mu_i \, \mathrm{d}m(g) \, .$$

Let ν be a weak-$*$-limit point of $(\nu_i)_{i \in I}$. Then the unique inner regular BOREL *probability representing $f \mapsto \int f \, \mathrm{d}\nu$ is α-invariant.*

Proof. W.l.o.g. $(\nu_i)_{i \in I}$ is weak-$*$-convergent to an inner regular BOREL probability ν. We use Proposition 1.13.1. Let $f \in \mathcal{C}_c(X, \mathbb{R})$ and $h \in G$. As f has compact support $|f|$ is bounded from above by some $C \in \mathbb{R}^+$.

The idea is now to transform $\int f \circ \alpha(h, \cdot) \, \mathrm{d}\nu$ into

$$\lim_{i \in I} \frac{1}{m(F_i)} \int_{h \cdot F_i} \int f \circ \alpha(hg, \cdot) \, \mathrm{d}\mu_i \, \mathrm{d}m(g) \, .$$

On the other hand $\int f \, \mathrm{d}\nu = \frac{1}{m(F_i)} \lim_{i \in I} \int_{F_i} \int f \alpha(hg, \cdot) \, \mathrm{d}\mu_i \, \mathrm{d}m(g)$. The difference of both integrals only depends on $\frac{m(F_i \triangle h F_i)}{m(F_i)}$ which converges to zero.

In detail:

$$\begin{aligned}
&\left| \int f \circ \alpha(h, \cdot) \, \mathrm{d}\nu - \int f \, \mathrm{d}\nu \right| \\
=& \lim_{i \in I} \left| \int f \circ \alpha(h, \cdot) \, \mathrm{d}\nu_i - \int f \, \mathrm{d}\nu_i \right| \\
=& \lim_{i \in I} \frac{1}{m(F_i)} \left| \int_{F_i} \int f \circ \alpha(h, \cdot) \circ \alpha(g, \cdot) \, \mathrm{d}\mu_i \, \mathrm{d}m(g) - \int f \, \mathrm{d}\nu_i \right| \\
=& \lim_{i \in I} \frac{1}{m(F_i)} \left| \int_{F_i} \int f \circ \alpha(h \cdot g, \cdot) \, \mathrm{d}\mu_i \, \mathrm{d}m(g) - \int_{F_i} \int f \circ \alpha(g, \cdot) \, \mathrm{d}\mu_i \, \mathrm{d}m(g) \right| \, .
\end{aligned}$$

[59] using the RIESZ Representation Theorem 1.8.17

Now we simplify and estimate

$$\begin{aligned}
&\lim_{i\in I}\frac{1}{m(F_i)}\left|\int_{F_i}\int f\circ\alpha(h\cdot g,\cdot)\ \mathrm{d}\mu_i\ \mathrm{d}m(g)-\int_{F_i}\int f\circ\alpha(g,\cdot)\ \mathrm{d}\mu_i\ \mathrm{d}m(g)\right|\\
=&\lim_{i\in I}\frac{1}{m(F_i)}\left|\int_{h\cdot F_i}\int f\circ\alpha(g,\cdot)\ \mathrm{d}\mu_i\ \mathrm{d}m(g)-\int_{F_i}\int f\circ\alpha(g,\cdot)\ \mathrm{d}\mu_i\ \mathrm{d}m(g)\right|\\
=&\lim_{i\in I}\frac{1}{m(F_i)}\left|\int_{h\cdot F_i\triangle F_i}\int f\circ\alpha(g,\cdot)\ \mathrm{d}\mu_i\ \mathrm{d}m(g)\right|\\
\leqslant&\lim_{i\in I}\frac{1}{m(F_i)}\int_{h\cdot F_i\triangle F_i}\int |f\circ\alpha(g,\cdot)|\ \mathrm{d}\mu_i\ \mathrm{d}m(g)\\
\leqslant&\lim_{i\in I}\frac{1}{m(F_i)}\int_{h\cdot F_i\triangle F_i}\int C\ \mathrm{d}\mu_i\ \mathrm{d}m(g)\\
=&\lim_{i\in I}\frac{1}{m(F_i)}\int_{h\cdot F_i\triangle F_i} C\ \mathrm{d}m(g)\\
=&\lim_{i\in I} C\cdot\frac{m(h\cdot F_i\triangle F_i)}{m(F_i)}=0\,.
\end{aligned}$$

□

Recall that the set of all probability measures on $\mathfrak{B}X$ is compact by the Theorem of Banach-Alaoglu 1.9.5.

Corollary 1.13.2.1. *There is at least one α-invariant probability μ on X.*

Proof. Apply Theorem 1.13.2 to $\mu_i := \delta_x$ for $x \in X$. By compactness of the space of probability measures, $(\nu_n)_{n\in\mathbb{N}}$ has a weak-$*$-convergent subnet with limit $\tilde{\nu}$. By the Riesz Representation Theorem 1.8.17 there is an inner regular Borel probability ν representing $f \mapsto \int f\ \mathrm{d}\tilde{\nu}$. □

1.13.2 Unique Ergodicity

The group G shall be locally compact and m be a left Haar measure. Let $(I,<)$ be a directed set and $\mathcal{F} = (F_i)_{i\in I}$ be an ergodic net. Fix any $(X, G, \alpha) \in \mathsf{CHausDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$. We have seen that $\mathbf{X}$ has an invariant probability.

Definition 1.13.3 (Unique Ergodicity)**.** We call $\mathbf{X}$ uniquely ergodic if and only if there is exactly one invariant probability.

Lemma 1.13.4. *If $\mathbf{X}$ is uniquely ergodic with invariant probability μ, then μ is ergodic.*

Proof. This is a direct consequence of Theorem 1.12.9. □

Theorem 1.13.5. *Let $\mathbf{X}$ be uniquely ergodic with invariant probability μ. Then*

$$\frac{1}{m(F_i)}\int_{F_i} f\circ\alpha(g,\cdot)\ \mathrm{d}m(g)\xrightarrow{i\in I}\int f\ \mathrm{d}\mu$$

in the uniform sense for any $f \in \mathcal{C}(X,\mathbb{R})$.

Proof. Assume the convergence is not uniform. Then there is $\varepsilon > 0$ such that for any $j \in I$ there is $i_j > j$ and x_j such that

$$\left| \frac{1}{m(F_{i_j})} \int_{F_{i_j}} f \circ \alpha(g, x_j) \, \mathrm{d}m(g) - \int f \, \mathrm{d}\mu \right| > \varepsilon \,. \tag{10}$$

Define a net of probabilities $(\nu_j)_{j \in I}$ by

$$\frac{1}{m(F_{i_j})} \int_{F_{i_j}} f \circ \alpha(g, x_j) \, \mathrm{d}m(g) = \frac{1}{m(F_{i_j})} \int_{F_{i_j}} f \circ \alpha(g,) \, \mathrm{d}\delta_{x_j} \, \mathrm{d}m(g) =: \int f \, \mathrm{d}\nu_j \,.$$

By compactness of the space of probabilities there is a convergent subnet of $(\nu_j)_{j \in I}$. By Theorem 1.13.2 the limit of every converging subnet must be invariant. Hence every converging subnet converges to μ. This means that $(\nu_j)_{j \in I}$ itself converges to μ. This is a contradiction to (10). □

Theorem 1.13.6. **X** *is uniquely ergodic if and only if the net of ergodic averages*

$$\frac{1}{m(F_i)} \int_{F_i} f \circ \alpha(g, x) \, \mathrm{d}m(g)$$

converges pointwise to a constant $C(f)$ for any $f \in \mathcal{C}(X, \mathbb{R})$.

Proof. Theorem 1.13.5 yields that the former implies the latter. Conversely, assume that **X** is not uniquely ergodic. Observe that

$$C : \mathcal{C}(X, \mathbb{R}) \longrightarrow \mathbb{R}, \; g \longmapsto C(g)$$

is a positive linear form. By the Riesz Representation Theorem 1.8.17 there is a probability μ such that $C(g) = \int g \, \mathrm{d}\mu$ for any $g \in \mathcal{C}(X, \mathbb{R})$. Observe that C, and thus μ, is invariant. As **X** is not uniquely ergodic, there is an ergodic probability ν such that $\nu \neq \mu$. By Corollary 1.8.17.1 there is $f \in \mathcal{C}(X, \mathbb{R})$ such that $\int f \, \mathrm{d}\mu \neq \int f \, \mathrm{d}\nu$. Let $\mathfrak{A}$ be the Borel-σ-algebra on X. By the Mean Ergodic Theorem 1.12.6 the net of ergodic averages $x \mapsto \frac{1}{m(F_i)} \int_{F_i} f \circ \alpha(g, x) \, \mathrm{d}m(g)$ converges in $L^1(\nu)$ to $\mathbb{E}_\nu [f \mid \mathfrak{I}_\nu]$. Here $\mathfrak{I}_\nu \leq \mathfrak{A}$ denotes the σ-algebra of ν-almost surely invariant sets. Now assume that $\frac{1}{m(F_i)} \int_{F_i} f \circ \alpha(g, x) \, \mathrm{d}m(g)$ converges pointwise to $C(f) = \int f \, \mathrm{d}\mu$. As X is compact f is bounded. The Lebesgue Convergence Theorem 1.6.21 implies that the net of ergodic averages converges in $L^1(\nu)$ to $\int f \, \mathrm{d}\mu$. This is a contradiction, so the ergodic averages can not converge pointwise to $C(f)$. □

1.13.3 Generic Points

Fix $(X, G, \alpha) \in \mathsf{CHausDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$. Assume that $\mathcal{F} = (F_i)_{i \in I}$ is a Følner net in G and μ an α-invariant probability on $\mathfrak{B}X$.

Definition 1.13.7 (Generic Point)**.** A point $x \in X$ is **generic** if and only if

$$\forall f \in \mathcal{C}(X, \mathbb{R}) : \lim_{i \in I} \int f \, \mathrm{d} \left(\frac{1}{m(F_i)} \int_{F_i} \delta_{\alpha(g,x)} \, \mathrm{d}m(g) \right) = \int f \, \mathrm{d}\mu \,. \tag{11}$$

Theorem 1.13.8. *Suppose there is at least one generic point x and the probability μ is ergodic. Then every open set of positive measure contains a generic point.*

Proof. Let E be the set of generic points. Note that E is invariant. As $\mathrm{cl}(E)$ is also invariant we see that $\mu(\mathrm{cl}(E)) \in \{0, 1\}$. Suppose $\mu(\mathrm{cl}(E)) = 0$. As $\mathrm{cl}(E)$ is closed we conclude by RIESZ Representation Theorem 1.8.17 that there is $f \in \mathcal{C}(X, \mathbb{R})$ with $\mathbb{1}_{\mathrm{cl}(E)} \leqslant f \leqslant \mathbb{1}_X$ such that

$$\int f \,\mathrm{d}\mu < 1\,.$$

Pick any $z \in E \subseteq \mathrm{cl}(E)$. By invariance of E we have that the probability

$$\eta_i := \frac{1}{F_i} \int_{F_i} \delta_{\alpha(g,z)} \,\mathrm{d}m(g)$$

is supported on $\mathrm{cl}(E)$ for $i \in I$. Thus $\int f \,\mathrm{d}\eta_i = 1$ and

$$\int f \,\mathrm{d}\mu = \lim_{i \in I} \int f \,\mathrm{d}\eta_i = 1\,.$$

A contradiction, thus $\mu(\mathrm{cl}(E)) = 1$. Now let U be open and $\mu(U) > 0$. First of all as $\mathrm{cl}(E)$ is full we know $U \cap \mathrm{cl}(E) \neq \emptyset$. If $U \cap E = \emptyset$, then U^c is a closed superset of E and thus $\mathrm{cl}(E) \subseteq U^c$. Thus U contains a generic point. □

Remark 1.13.8.1. The theorem shows that the generic points are dense in the support of an ergodic probability. As there can be multiple ergodic probabilities even in minimal systems we also obtain two corollary results

i) The set of generic points is in general not closed.

ii) Generally not every point from the support is generic.

2 The Maximal Equicontinuous Factor

Definition 2.0.1 (Maximal Equicontinuous Factor)**.** Fix $(X, G, \alpha) \in \mathsf{CHausDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$. A pair $(\mathbf{Y}, \pi^{\mathbf{X}}_{\mathtt{mef}})$ consisting of $\mathbf{Y} := (Y, G, \beta) \in \mathsf{EquiDyn}(G)$ and a factor map $\pi^{\mathbf{X}}_{\mathtt{mef}} : \mathbf{X} \to \mathbf{Y}$ is called a **maximal equicontinuous factor** (`mef`)[60] of $\mathbf{X}$ if and only if for any equicontinuous system $\mathbf{Z} = (Z, G, \gamma)$ and each factor map $\pi^{\mathbf{X}}_{\mathbf{Z}} : \mathbf{X} \to \mathbf{Z}$ there is a unique factor map $\pi^{\mathtt{mef}}_{\mathbf{Z}} : \mathbf{Y} \to \mathbf{Z}$ such that $\pi^{\mathbf{X}}_{\mathbf{Z}} = \pi^{\mathtt{mef}}_{\mathbf{Z}} \circ \pi^{\mathbf{X}}_{\mathtt{mef}}$.

2.1 Universal Factor Maps

Fix any $(X, G, \alpha) \in \mathsf{CHausDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$. Let $(\mathbf{Y}, \pi^{\mathbf{X}}_{\mathtt{mef}})$ be a `mef` of $\mathbf{X}$.

While the factor map $\pi^{\mathbf{X}}_{\mathtt{mef}}$ plays a crucial role in Definition 2.0.1 it is common practice to notationally suppress it and call $\mathbf{Y}$ a `mef`. Before we state a general existence and uniqueness result we will explore for which factor maps $\pi : \mathbf{X} \to \mathbf{Y}$ the pair $(\mathbf{Y}, \pi)$ is a `mef` of $\mathbf{X}$. For that endeavour a little technical apparatus is useful:

[60] pronounced [MεF]

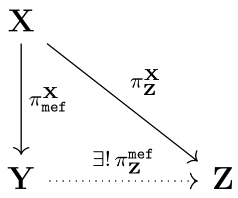

Figure 5: Universal Property of a `mef`

Definition 2.1.1. The **endomorphism monoid** of $\mathbf{X}$ is given by

$$\operatorname{End}(\mathbf{X}) := (\{f \in \operatorname{End}(X) \mid f \text{ is equivariant}\}, \circ) \ ,$$

where $\operatorname{End}(X)$ is the monoid of surjective continuous mappings.

Definition 2.1.2. The **automorphism group** of $\mathbf{X}$ is given by

$$\operatorname{Aut}(\mathbf{X}) := (\{f \in \operatorname{Homeo}(X) \mid f \text{ is equivariant}\}, \circ) \ ,$$

where $\operatorname{Homeo}(X)$ denotes the group of all self-homeomorphisms of X.

Definition 2.1.3 (Action on factor maps)**.** Define a monoid action

$$\begin{aligned} C : \operatorname{End}(\mathbf{Y}) \times \operatorname{Hom}_{\mathsf{CHausDyn}(G)}(\mathbf{X}, \mathbf{Y}) &\longrightarrow \operatorname{Hom}_{\mathsf{CHausDyn}(G)}(\mathbf{X}, \mathbf{Y}) \\ (e, \pi) &\longmapsto e \circ \pi \ . \end{aligned}$$

By restricting to $\operatorname{Aut}(\mathbf{Y})$ we obtain a group action.

Proposition 2.1.4. *The monoid action C is faithful. Furthermore, the orbit of $\pi^{\mathbf{X}}_{mef}$ is the whole set* $\operatorname{Hom}_{\mathsf{CHausDyn}(G)}(\mathbf{X}, \mathbf{Y})$.

Proof. The proof works by inspecting the Universal Property 5. Let $\rho : \mathbf{X} \to \mathbf{Y}$ be any factor map. As $\mathbf{Y}$ is equicontinuous there must be a unique factor map $a : \mathbf{Y} \to \mathbf{Y}$ such that $\rho = a \circ \pi^{\mathbf{X}}_{\mathtt{mef}}$. Thus the orbit of $\pi^{\mathbf{X}}_{\mathtt{mef}}$ is the whole set $\operatorname{Hom}_{\mathsf{CHausDyn}(G)}(\mathbf{X}, \mathbf{Y})$.

If $a \neq b$, there is $y \in \mathbf{Y}$ such that $a(y) \neq b(y)$. $\rho : \mathbf{X} \to \mathbf{Y}$ is surjective like all factor maps. Thus, there is $x \in \mathbf{X}$ such that $\rho(x) = y$. Then $a \circ \rho(x) \neq b \circ \rho(x)$. Thus C is faithful. □

Remark 2.1.4.1. Note that unlike in the case of group actions the fact that one orbit of a monoid action is the whole set does not imply that all orbits of the monoid action are the whole set.

Let us inspect further consequences of the Universal Property 5. For any $a \in \operatorname{End}(\mathbf{X})$ the map $\pi^{\mathbf{X}}_{\mathtt{mef}} \circ a$ is a factor map to an equicontinuous system. Thus there must be $b_a \in \operatorname{End}(\mathbf{Y})$ such that $b_a \circ \pi^{\mathbf{X}}_{\mathtt{mef}} = \pi^{\mathbf{X}}_{\mathtt{mef}} \circ a$. This element b_a is uniquely determined. This yields

$$\pi^{\mathbf{X}}_{\mathtt{mef}} \circ (a_1 \circ a_2) = b_{(a_1 \circ a_2)} \circ \pi^{\mathbf{X}}_{\mathtt{mef}}$$

and also

$$
\begin{aligned}
(\pi^{\mathbf{X}}_{\texttt{mef}} \circ a_1) \circ a_2 &= (b_{a_1} \circ \pi^{\mathbf{X}}_{\texttt{mef}}) \circ a_2 \\
&= b_{a_1} \circ (\pi^{\mathbf{X}}_{\texttt{mef}} \circ a_2) \\
&= b_{a_1} \circ b_{a_2} \circ \pi^{\mathbf{X}}_{\texttt{mef}}
\end{aligned}
$$

Thus, by associativity, the mapping $a \mapsto b_a$ is a monoid homomorphism

$$\mathrm{End}(\pi^{\mathbf{X}}_{\texttt{mef}}) : \mathrm{End}(\mathbf{X}) \to \mathrm{End}(\mathbf{Y})\,.$$

The fact that $\mathrm{End}(\pi^{\mathbf{X}}_{\texttt{mef}})(\mathrm{Id}_{\mathbf{X}}) = \mathrm{Id}_{\mathbf{Y}}$ follows again from uniqueness and

$$\pi^{\mathbf{X}}_{\texttt{mef}} \circ \mathrm{Id}_{\mathbf{X}} = \pi^{\mathbf{X}}_{\texttt{mef}} = \mathrm{Id}_{\mathbf{Y}} \circ \pi^{\mathbf{X}}_{\texttt{mef}}\,.$$

As invertible elements get mapped to invertible elements we further obtain a group homomorphism

$$\mathrm{Aut}(\pi^{\mathbf{X}}_{\texttt{mef}}) : \mathrm{Aut}(\mathbf{X}) \to \mathrm{Aut}(\mathbf{Y})\,.$$

Those homomorphisms extend the mapping

$$\alpha(g,\cdot) \longmapsto \beta(g,\cdot)\,,$$

where β is the dynamic on the `mef`.

Lemma 2.1.5. *The induced homomorphisms have the kernel*

$$
\begin{aligned}
\ker\big(\mathrm{End}(\pi^{\mathbf{X}}_{mef})\big) &= \big\{a \in \mathrm{End}(\mathbf{X}) \;\big|\; \pi^{\mathbf{X}}_{mef} \circ a = \pi^{\mathbf{X}}_{mef}\big\} \\
\ker\big(\mathrm{Aut}(\pi^{\mathbf{X}}_{mef})\big) &= \big\{a \in \mathrm{Aut}(\mathbf{X}) \;\big|\; \pi^{\mathbf{X}}_{mef} \circ a = \pi^{\mathbf{X}}_{mef}\big\}
\end{aligned}
$$

Definition 2.1.6 (Universality)**.** We say that a factor map $\rho : \mathbf{X} \to \mathbf{Y}$ is **universal** if and only if ρ satisfies the Universal Property 5, i.e. $(\mathbf{Y}, \rho)$ is a `mef` of $\mathbf{X}$.

Theorem 2.1.7. *A factor map $\rho : \mathbf{X} \to \mathbf{Y}$ is universal if and only if there is $a \in \mathrm{Aut}(\mathbf{Y})$ such that $\rho = a \circ \pi^{\mathbf{X}}_{mef}$.*

Proof. Again we simply unwrap the Universal Property 5.

Suppose that ρ is universal. As $\rho : \mathbf{X} \to \mathbf{Y}$ is a factor map to an equicontinuous system, there is an $a \in \mathrm{End}(\mathbf{Y})$ such that $\rho = a \circ \pi^{\mathbf{X}}_{\texttt{mef}}$. Now as ρ is universal, there is another $b \in \mathrm{End}(\mathbf{Y})$ such that $\pi^{\mathbf{X}}_{\texttt{mef}} = b \circ \rho$. We see $\rho = a \circ b \circ \rho$. As ρ is surjective we conclude $a \circ b = \mathrm{Id}_{\mathbf{Y}}$. Thus $a \in \mathrm{Aut}(\mathbf{Y})$.

Now suppose that there is $b \in \mathrm{Aut}(\mathbf{Y})$ such that $\pi^{\mathbf{X}}_{\texttt{mef}} = b \circ \rho$. Let $\mathbf{Z}$ be any equicontinuous system and $q : \mathbf{X} \to \mathbf{Z}$ be a factor map. As $\pi^{\mathbf{X}}_{\texttt{mef}}$ is universal there is a unique map $\pi^{\texttt{mef}}_{\mathbf{Z}}$ such that $q = \pi^{\texttt{mef}}_{\mathbf{Z}} \circ \pi^{\mathbf{X}}_{\texttt{mef}}$. Note that $q = \pi^{\texttt{mef}}_{\mathbf{Z}} \circ b \circ \rho$. That proves the existence part of the Universal Property 5. Now let $r : \mathbf{Y} \to \mathbf{Z}$ be arbitrary such that $q = r \circ \rho$. Then $q = (r \circ b) \circ \pi^{\mathbf{X}}_{\texttt{mef}}$. However, due to the universal property of $\pi^{\mathbf{X}}_{\texttt{mef}}$ again, the mapping $r \circ b$ is uniquely determined by this equation. As b is an automorphism, this means that r is unique as well. That proves the uniqueness part of the Universal Property 5. □

Corollary 2.1.7.1. *A factor map $\rho : \mathbf{X} \to \mathbf{Y}$ is universal if and only if the orbit of ρ under the monoid action C of* $\mathrm{End}(\mathbf{Y})$ *is the whole set of factor maps* $\mathrm{Hom}_{\mathsf{CHausDyn}(G)}(\mathbf{X}, \mathbf{Y})$.

Proof. If ρ satisfies the Universal Property 5 then we can repeat the above argumentation for ρ and obtain that it has full orbit.

Conversely assume that the orbit of ρ is the whole set $\mathrm{Hom}_{\mathsf{CHausDyn}(G)}(\mathbf{X}, \mathbf{Y})$. In particular there is $b \in \mathrm{End}(\mathbf{Y})$ such that $b \circ \rho = \pi^{\mathbf{X}}_{\mathtt{mef}}$. Furthermore, there is as always an $a \in \mathrm{End}(\mathbf{Y})$ such that $\rho = a \circ \pi^{\mathbf{X}}_{\mathtt{mef}}$. We conclude $\rho = (a \circ b) \circ \rho$. By the uniqueness part of the Universal Property 5 we conclude $a \circ b = \mathrm{Id}_{\mathbf{Y}} = b \circ a$. So a is invertible, i.e. $a \in \mathrm{Aut}(\mathbf{Y})$. By Theorem 2.1.7 we conclude that ρ is universal. □

Question 2.1.8. *Are all factor maps $\pi : \mathbf{X} \to \mathbf{Y}$ universal?*

By Proposition 2.1.7 together with 2.1.4 it is sufficient to show that any endomorphism of equicontinuous systems is an automorphism, i.e. $\mathrm{End}(\mathbf{Y}) = \mathrm{Aut}(\mathbf{Y})$. Thus we ask

Question 2.1.9. *Is any endomorphism of equicontinuous systems already an automorphism?*

For a special case we can answer those questions positively:

Theorem 2.1.10. *Let the group G be Abelian and let $\mathbf{X} = (X, G, \alpha)$ be minimal. Then all factor maps $\rho : \mathbf{X} \to \mathbf{Y}$ are universal.*

Theorem 2.1.10 follows from a combination of the Classification Theorem 4.4.1 presented later, which asserts that (minimal) Abelian equicontinuous systems are conjugate to group rotations, and the following

Theorem 2.1.11. *Let $\psi : G \to H$ be a group compactification. Define the system $\mathbf{X} = \mathsf{Rot}(H, \psi)$, i.e. $\mathbf{X} = (H, G, \alpha)$ where*

$$\alpha(g, h) = \psi(g) \cdot h\,.$$

Then $\mathrm{End}(\mathbf{X}) = \mathrm{Aut}(\mathbf{X}) \cong H$.

Proof. Let $\varphi \in \mathrm{End}(\mathbf{X})$. Then for $h \in H$ and $g \in G$ we have

$$\varphi(\psi(g)h) = \varphi(\alpha(g, h)) = \alpha(g, \varphi(h)) = \psi(g)\varphi(h)$$

by equivariance. For $f \in H$ define $\gamma_f : H \to H$, $h \mapsto h \cdot f$. Calculate

$$\gamma_f(\alpha(g, h)) = \alpha(g, h)f = \psi(g)hf = \alpha(g, \gamma_f(h))\,.$$

So $\gamma_f \in \mathrm{End}(\mathbf{X})$. As $(\gamma_f)^{-1} = \gamma_{f^{-1}}$ we have $\gamma_f \in \mathrm{Aut}(\mathbf{X})$. Let $f := \varphi(1_H)$ and conclude $\gamma_f(1_H) = \varphi(1_H)$. By equivariance we see that γ_f and φ must coincide on the whole orbit of 1_H under the action of $\mathrm{End}(\mathbf{X})$. Clearly the orbit of 1_H contains the dense set $\psi(G)$. As both φ and γ_f are continuous they must be equal on the whole group H. So $\varphi = \gamma_f \in \mathrm{Aut}(\mathbf{X})$. □

Remark 2.1.11.1. a) Note that interestingly we can use any $\psi(G)$ as an automorphism and not just elements from the center $\mathrm{Z}(G)$. This is due to the fact that G acts by left-multiplication and the automorphisms by multiplication from the right. In fact, for any group H and $g, h, f \in H$ by associativity

$$(gh)f = ghf = g(hf)\,.$$

So multiplication from the left and from the right commute by associativity (and in particular Abelianity is not needed here)

b) The theorem cannot easily be generalized to actions on the homogeneous space G/F for a (closed) subgroup F. This is due the fact that if F is not normal (i.e. G/F is not group itself) the right multiplication is generally not well-defined on left cosets. For example, the map $hF \mapsto hfF$ is not independent of the choice of the representative h for arbitrary $f \in G$.

2.2 Existence and Uniqueness

Theorem 2.2.1. *Every* $\mathbf{X} := (X, G, \alpha) \in \mathsf{CHausDyn}(G)$ *has a* `mef` $(\mathbf{Y}, \pi)$*. If* $(\mathbf{Z}, \rho)$ *is another* `mef`*, then there is a conjugacy* $\eta : \mathbf{Y} \to \mathbf{Z}$ *such that* $\rho = \eta \circ \pi$*.*

Proof of uniqueness. As ρ is a factor map into an equicontinuous system there must be a unique map $\eta : \mathbf{Y} \to \mathbf{Z}$ such that $\eta \circ \pi = \rho$. Similarly there must be a unique $\overline{\eta} : \mathbf{Z} \to \mathbf{Y}$ such that $\overline{\eta} \circ \rho = \pi$. Clearly $(\overline{\eta} \circ \eta) \circ \pi = \pi$ and $\mathrm{Id}_{\mathbf{Y}} \circ \pi = \pi$. By the uniqueness in the Universal Property 5 we learn that $\overline{\eta} \circ \eta = \mathrm{Id}_{\mathbf{Y}}$. Similarly $\eta \circ \overline{\eta} = \mathrm{Id}_{\mathbf{Z}}$. So η is indeed a conjugacy. □

A constructive proof for the existence can be found in [Auslander, 1988, Chapter 9, Theorem 1]. We present the core ideas of the construction: The central notion is the one of an `icer` defined by

Definition 2.2.2 (`icer`)**.** A relation $\sim \subseteq X \times X$ is called an **invariant closed equivalence relation** (`icer`) if and only if

(1) $\sim$ is invariant, i.e. $\forall g \in G : \alpha(g, x) \sim \alpha(g, y) \iff x \sim y$,

(2) $\sim$ is closed as a subset of the product space $X \times X$,

(3) $\sim$ is an equivalence relation.

There is a correspondence between factor maps and `icer`. Given an `icer` $\sim$ we can consider

$$\mathbf{X}/{\sim} := (X/{\sim}, G, \alpha/{\sim})\,,$$

where $\alpha/{\sim}$ is given as

$$\alpha/{\sim}\,(g, [x]_\sim) = [\alpha(g, x)]_\sim\,.$$

Here $[y]_\sim \in X/{\sim}$ denotes the equivalence class of $y \in X$. Note that $\alpha/{\sim}$ is well-defined by the invariance of $\sim$. In particular the projection $\pi : X \to X/{\sim}$ yields a factor map.

Conversely, we can assign to any factor map $\pi : \mathbf{X} \to \mathbf{Z}$ an `icer` given by

$$\sim_\pi := \{(x,y) \in X \times X \mid \pi(x) = \pi(y)\} \ .$$

If one considers the system of `icer` of $\mathbf{X}$ and the system of its factors as categories, then those correspondences turn out to be essentially inverse functors[61], establishing an equivalence of categories.

Sketch of proof for existence of a `mef`. The main idea is that the maximality of a factor (map) translates into the minimality of the corresponding `icer`. In other words, we look for the smallest `icer` (w.r.t. inclusion) corresponding to an equicontinuous factor. Let

$$E := \{\sim \subseteq X \times X \mid \sim \text{ is } \texttt{icer} \text{ and } \mathbf{X}/\!\sim \text{ is equicontinuous}\}$$

be the system of all such `icer`. In the realm of `icer` one can simply take the intersection $\sim^* := \bigcap E$. The result $\sim^*$ is clearly again an `icer`. It remains to show that $\sim^* \in E$, i.e. $\mathbf{X}/\!\sim^*$ is equicontinuous.

This can be seen by realizing that $\mathbf{X}/\!\sim^*$ is conjugate to a closed subsystem of

$$\prod_{\sim \in E} \mathbf{X}/\!\sim \ .$$

Clearly taking products and restricting to closed subsystems preserves equicontinuity.

It remains to verify the Universal Property in Figure 5. Let $\pi_{\mathbf{Z}}^{\mathbf{X}} : \mathbf{X} \to \mathbf{Z}$ be a factor map where $\mathbf{Z}$ is equicontinuous. There is an `icer` $\sim \in E$ such that $\mathbf{X}/\!\sim$ is conjugate to $\mathbf{Z}$. Let $\Psi : \mathbf{X}/\!\sim \to \mathbf{Z}$ be this conjugacy. Then $\Psi^{-1} \circ \pi_{\mathbf{Z}}^{\mathbf{X}} : \mathbf{X} \to \mathbf{X}/\!\sim$ is a factor map. Clearly $\sim^* \subseteq \sim$. Thus $\Psi^{-1} \circ \pi_{\mathbf{Z}}^{\mathbf{X}} : X \to X/\!\sim$ is constant on the equivalence classes of $\sim^*$. Of course this also holds for $\pi_{\mathbf{Z}}^{\mathbf{X}}$.

This means that the map

$$\pi_{\mathbf{Z}}^{\texttt{mef}} : X/\!\sim^* \longrightarrow Z, \ \pi_{\mathbf{Z}}^{\texttt{mef}}([x]_{\sim^*}) = \pi_{\mathbf{Z}}^{\mathbf{X}}(x)$$

is well-defined. Obviously $\pi_{\mathbf{Z}}^{\texttt{mef}}$ satisfies the existence part of Universal Property 5. It is easy to see that it is the only function doing so. □

Proposition 2.2.3. *Let X be metrizable and $\sim \subseteq X \times X$ be an equivalence relation. Then $X/\!\sim$ is metrizable if and only if $\sim$ is closed.*[62]

We conclude

Proposition 2.2.4. *Let $(X, G, \alpha) = \mathbf{X}$ be a `tds`. If X is compact and metrizable then any `mef` is also compact and metrizable.*

[61] This means that they are inverse up to isomorphy.

[62] This is a result in my bachelor's thesis [Haupt, 2020, Theorem 1.29].

2.3 Functoriality

Theorem 2.2.1 provides a mapping `mef` from the class of objects $\mathrm{ob}(\mathsf{CHausDyn}(G))$ to the subclass $\mathrm{ob}(\mathsf{EquiDyn}(G))$, assigning each system its `mef`. It turns out, that the mapping `mef` can be turned into a functor MEF.

Theorem 2.3.1. *There is a functor*

$$\mathsf{MEF} : \mathsf{CHausDyn}(G) \longrightarrow \mathsf{EquiDyn}(G)$$

sending each `tds` $\mathbf{X}$ *to* `mef`$(\mathbf{X})$.

Remark 2.3.1.1. The functoriality of MEF implies that commutative diagrams are preserved. In particular if $\mathbf{Z}$ is a factor of $\mathbf{X}$ then $\mathsf{MEF}(\mathbf{Z})$ is a factor of $\mathsf{MEF}(\mathbf{X})$.

Constructive Proof. The construction in Theorem 2.2.1 yields a `mef` $(\mathsf{MEF}(\mathbf{X}), \pi^{\mathbf{X}}_{\texttt{mef}})$, given by a factoring out an `icer`, for each $\mathbf{X} \in \mathsf{CHausDyn}(G)$. It remains to define what MEF does to factor maps. Let $\mathbf{X}, \mathbf{Y} \in \mathsf{CHausDyn}(G)$ and $\Psi : \mathbf{X} \to \mathbf{Y}$ be a factor map. Then

$$\psi := \pi^{\mathbf{Y}}_{\texttt{mef}} \circ \Psi : \mathbf{X} \longrightarrow \mathsf{MEF}(\mathbf{Y})$$

is a factor map from $\mathbf{X}$ to an equicontinuous system. By the Universal Property 5 ψ must factor through $\pi^{\mathbf{X}}_{\texttt{mef}}$, i.e. there is a unique $\eta : \mathsf{MEF}(\mathbf{X}) \to \mathsf{MEF}(\mathbf{Y})$ such that $\eta \circ \pi^{\mathbf{X}}_{\texttt{mef}} = \psi$. We define $\mathsf{MEF}(\Psi) := \eta$. See Figure 2.3 for a visualization of that construction. Let $\mathbf{Z} \in \mathsf{CHausDyn}(G)$ be yet another `tds` and $\Theta : \mathbf{Y} \to \mathbf{Z}$ a factor

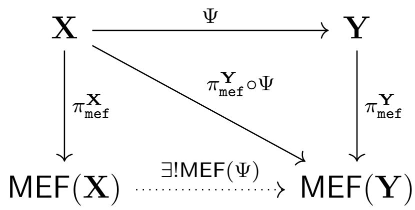

Figure 6: Construction of $\mathsf{MEF}(\Psi)$

map. We must show that

$$\mathsf{MEF}(\Theta \circ \Psi) = \mathsf{MEF}(\Theta) \circ \mathsf{MEF}(\Psi)\,.$$

Recall that $\mathsf{MEF}(\Psi) \circ \pi^{\mathbf{X}}_{\texttt{mef}} = \pi^{\mathbf{Y}}_{\texttt{mef}} \circ \Psi$ and $\mathsf{MEF}(\Theta) \circ \pi^{\mathbf{Y}}_{\texttt{mef}} = \pi^{\mathbf{Z}}_{\texttt{mef}} \circ \Theta$. Therefore

$$\begin{aligned} \mathsf{MEF}(\Theta) \circ \mathsf{MEF}(\Psi) \circ \pi^{\mathbf{X}}_{\texttt{mef}} &= \mathsf{MEF}(\Theta) \circ \pi^{\mathbf{Y}}_{\texttt{mef}} \circ \Psi \\ &= \pi^{\mathbf{Z}}_{\texttt{mef}} \circ \Theta \circ \Psi\,. \end{aligned}$$

So we have

$$\mathsf{MEF}(\Theta) \circ \mathsf{MEF}(\Psi) \circ \pi^{\mathbf{X}}_{\texttt{mef}} = \pi^{\mathbf{Z}}_{\texttt{mef}} \circ \Theta \circ \Psi \tag{12}$$

as well as

$$\mathsf{MEF}(\Theta \circ \Psi) \circ \pi^{\mathbf{X}}_{\texttt{mef}} = \pi^{\mathbf{Z}}_{\texttt{mef}} \circ \Theta \circ \Psi\,. \tag{13}$$

By the uniqueness part of the Universal Property 5 we conclude from (12) and (13) that $\mathsf{MEF}(\Theta) \circ \mathsf{MEF}(\Psi) = \mathsf{MEF}(\Theta \circ \Psi)$. □

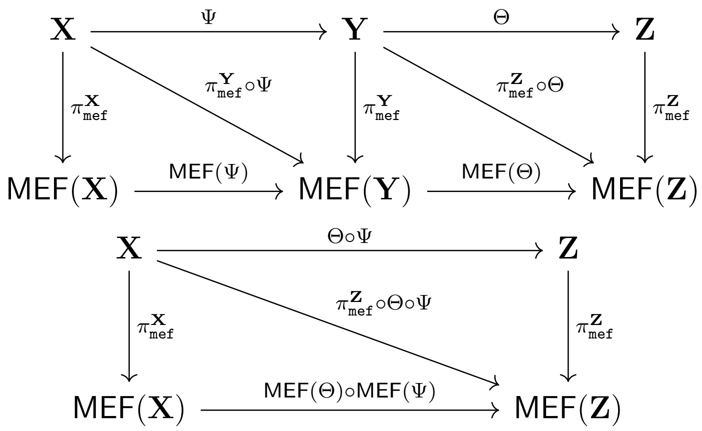

Figure 7: Functoriality of MEF

Remark 2.3.1.2. Note that the construction depends on the choice of the mefs $(\mathsf{MEF}(\mathbf{X}), \pi^{\mathbf{X}}_{\mathtt{mef}})$. In our case such a choice has been done by the general construction in Theorem 2.2.1. For any other choices (say by AXIOM OF CHOICE) we obtain potentially a different functor. However, we will see in Remark 2.3.2.1 that all arising functors will be naturally isomorphic.

Let U be the forgetful functor $U : \mathsf{EquiDyn}(G) \to \mathsf{CHausDyn}(G)$. This functor sends each system in the category $\mathsf{EquiDyn}(G)$ to the same system in the more general category $\mathsf{CHausDyn}(G)$. Despite being the same system for all dynamical purposes, the systems $\mathbf{X} \in \mathsf{EquiDyn}(G)$ and $U(\mathbf{X}) \in \mathsf{CHausDyn}(G)$ are *category-theoretically* fundamentally different. So formally we should have written the Universal Property 5 as in Figure 8.

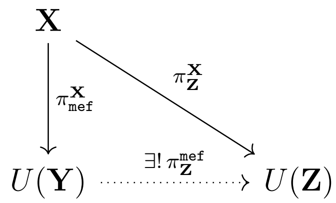

Figure 8: Universal Property of a mef with forgetful functor

Proposition 2.3.2. *Any mef $(\mathbf{Y}, \pi^{\mathbf{X}}_{mef})$ of a system $\mathbf{X} \in \mathsf{CHausDyn}(G)$ is a universal morphism from $\mathbf{X}$ to U.*

Proof. Compare Figures 8 and 4. □

Corollary 2.3.2.1. *The functor* MEF *is a left-adjoint of U.*

Proof. Relate the construction in Theorem 2.3.1 with the Construction 1.2.11. Then use Proposition 1.2.12. □

Remark 2.3.2.1. a) By Proposition 1.2.9 all left-adjoints of a functor are naturally isomorphic. So while a priori the construction in Theorem 2.3.1 depends on the choice of the factor maps $\pi^{\mathbf{X}}_{\mathtt{mef}}$, all resulting functors are naturally isomorphic.

b) By [Leinster, 2014, Theorem 6.3.1] the fact that the functor is a left-adjoint implies that co-limits[63] are preserved. The paradigmatic co-limit is the co-product $\coprod$.[64] The co-product in the categories of dynamical systems is the disjoint union. Thus, if a dynamical system $\mathbf{X}$ can be decomposed into closed subsystems, e.g. $\mathbf{X} = \coprod_{i \in I} \mathbf{X}_i$, one can treat them separately while calculating $\mathsf{MEF}(\mathbf{X})$, e.g. $\mathsf{MEF}(\mathbf{X}) = \coprod_{i \in I} \mathsf{MEF}(\mathbf{X}_i)$.

There are more abstract ways of obtaining the functor MEF. In [Delvenne, 2019, Theorem A2] the authors use the General Adjoint Functor Theorem[65] from Category Theory in order to prove the existence of a left-adjoint of the forgetful functor $U : \mathsf{EquiDyn}(G) \to \mathsf{CHausDyn}(G)$. They proceed to show that this is the functor MEF.

Proposition 2.3.3. *For any* $\mathbf{X} \in \mathsf{CHausDyn}(G)$ *we have*

$$\mathsf{MEF}(\mathbf{Z}^{\mathsf{op}}) \cong \mathsf{MEF}(\mathbf{Z})^{\mathsf{op}}.$$

Further if $\pi : \mathbf{X} \to \mathsf{MEF}(\mathbf{X})$ *is universal so is* $\pi^{\mathsf{op}} : \mathbf{X}^{\mathsf{op}} \to \mathsf{MEF}(\mathbf{X}^{\mathsf{op}})$.

Sketch of proof. As $U_{\mathsf{CHausDyn}(G)}(\pi^{\mathsf{op}}) = U_{\mathsf{CHausDyn}(G)}(\pi)$ the functor op preserves factor maps. Further op sends equicontinuous systems to equicontinuous systems.

Recall that we can characterize the system $\mathsf{MEF}(\mathbf{X})$ up to conjugacy by the Universal Property 5. This universal property only depends on the structure of factor maps from $\mathbf{X}$ into equicontinuous systems. This structure however is preserved by the above observations. □

3 The Hierarchy of Invertibility Properties

Let $(X, G, \alpha), (Y, G, \beta) \in \mathsf{CHausDyn}(G)$. Denote $\mathbf{X} := (X, G, \alpha)$ and $\mathbf{Y} := (Y, G, \beta)$. Let $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. π is a quotient map like any continuous surjection between compact Hausdorff spaces. In the case of (Y, π) being a `mef` of $\mathbf{X}$ this directly follows from the construction of a `mef` as a quotient space without invoking general arguments.

There are plenty of interesting additional properties one can impose on π. Those will strengthen the relation between the systems $\mathbf{X}$ and $\mathbf{Y}$. In the case of $(\mathbf{Y}, \pi)$ being a `mef` of $\mathbf{X}$ there will be an intimate correspondence between invertibility properties of the factor map π and the dynamical properties of $\mathbf{X}$.

Homeomorphic $\Longrightarrow$ almost surely 1-1 $\Longrightarrow$ Topo-Isomorphic & Almost 1-1 $\Longrightarrow$ Topo-Isomorphic

Figure 9: Hierarchy of Invertibility Properties

[63] See Definition 5.2.1. in [Leinster, 2014, p.126].
[64] The term co-product is defined in Definition 5.2.2. in [Leinster, 2014, p.127].
[65] See [Leinster, 2014, Theorem 6.3.10]

3.1 General Facts about Factor Maps

Lemma 3.1.1. *When the systems* $\mathbf{X}$*,* $\mathbf{Y}$ *are minimal and* $\pi : \mathbf{X} \to \mathbf{Y}$ *is a factor map the following dichotomy holds: Given* $a_1, a_2 \in \mathrm{End}(\mathbf{X})$ *we have*

$$\textbf{\textit{either}} \quad \forall x : \pi(a_1(x)) = \pi(a_2(x)) \quad \textbf{\textit{or}} \quad \forall x : \pi(a_1(x)) \neq \pi(a_2(x))\,.$$

In particular we have for $\pi = \mathrm{Id}_{\mathbf{X}}$

$$\textbf{\textit{either}} \quad \forall x : a_1(x) = a_2(x) \quad \textbf{\textit{or}} \quad \forall x : a_1(x) \neq a_2(x)\,.$$

Proof. Suppose that there is a point x such that $\pi(a_1(x)) = \pi(a_2(x))$. Then by equivariance we obtain that $\forall g \in G : \pi(a_1(\alpha(g,x))) = \pi(a_2(\alpha(g,x)))$. As the diagonal in $Y \times Y$ is closed due to the Hausdorff property $\pi \circ a_1$ and $\pi \circ a_2$ must coincide on a closed subset. By minimality this must be the whole space. ☐

Consider the map that assigns each fiber its diameter, which is given by

$$\pi_* \mathsf{diam} : Y \longrightarrow [0, \mathsf{diam}(X)]\,,\ y \longmapsto \mathsf{diam}\left(\pi^{-1}\{y\}\right)\,.$$

Lemma 3.1.2. $\pi_* \mathsf{diam}$ *is upper semi-continuous.*

Proof. We know that $y \mapsto \pi^{-1}(\{y\})$ is upper hemi-continuous by Theorem 1.5.9. The result follows from Lemma 1.5.10. ☐

Lemma 3.1.3. *Let* X *be metrizable with metric* d*. Further assume* $\mathbf{X}$ *and* $\mathbf{Y}$ *to be minimal. Then for any* $\delta > 0$ *there is* $\eta > 0$ *such that for any* $x \in X$ *there is* $y \in Y$ *such that* $B_\eta(y) \subseteq \pi(B_\delta(x))$*.*[66]

Proof. Let $\delta > 0$ and fix $x \in X$. As $\mathbf{X}$ is minimal there are finitely many $\{g_1, \ldots, g_n\}$ such that $X = \bigcup_{i=1}^n \alpha(g_i, B_\delta(x))$. One sees that

$$\begin{aligned} Y = \pi(X) &= \pi\left(\bigcup_{i=1}^n \alpha(g_i, B_\delta(x))\right) \\ &= \bigcup_{i=1}^n \pi\left(\alpha(g_i, B_\delta(x))\right) = \bigcup_{i=1}^n \beta\left(g_i, \pi\left(B_\delta(x)\right)\right)\,. \end{aligned}$$

Due to the Baire Category Theorem 1.7.5, at least one of the $\beta\left(g_i, \pi\left(B_\delta(x)\right)\right)$ must have non-empty interior. As $\beta(g, \cdot)$ is an homeomorphism we see that for any $g \in G$ the set $\beta\left(g, \pi\left(B_\delta(x)\right)\right)$ has non-empty interior. In particular $\pi\left(B_\delta(x)\right)$ must have non-empty interior.

There are finitely many $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^n B_{\frac{\delta}{2}}(x_i)\,.$$

[66] This is a generalization of the statement of Lemma 3.7 and of its proof as presented in the paper [García-Ramos et al., 2021].

For all $i \in \{1, \ldots, n\}$ there is $\eta_i > 0$ and $y_i \in Y$ such that

$$B_{\eta_i}(y_i) \subseteq \pi\left(B_{\frac{\delta}{2}}(x_i)\right).$$

Let $\eta = \min\{\eta_i \mid i \in \{1, \ldots, n\}\}$. Take any $x \in X$. There is an $i \in \{1, \ldots, n\}$ such that $x \in B_{\frac{\delta}{2}}(x_i)$. The triangle inequality yields that $B_{\frac{\delta}{2}}(x_i) \subseteq B_\delta(x)$. So

$$B_\eta(y_i) \subseteq B_{\eta_i}(y_i) \subseteq \pi\left(B_{\frac{\delta}{2}}(x_i)\right) \subseteq \pi\left(B_\delta(x)\right). \qquad \square$$

Lemma 3.1.4. *Let X and Y metrizable with metrics d_X and d_Y. Let $\pi : X \to Y$ be a continuous surjection and $y_0 \in Y$ such that* $\mathsf{card}\left(\pi^{-1}\left(\{y_0\}\right)\right) = 1$. *Then for any $\varepsilon > 0$ there is an $\eta > 0$ such that*

$$\forall y \in B_\eta(y_0) : \mathsf{diam}\left(\pi^{-1}\left(B_\eta(y)\right)\right) < \varepsilon.$$

Proof. Note that the multi-valued map

$$D_\rho : Y \twoheadrightarrow Y : \ y \longmapsto \{y' \in Y \mid d_Y(y, y') \leqslant \rho\}$$

has a closed graph for any $\rho \geqslant 0$ by continuity of d_Y. Thus by Theorem 1.5.9 we know that D_ρ is upper hemi-continuous. In turn $\pi^{-1} \circ D_\rho$ is upper hemi-continuous. Finally by Lemma 1.5.10 $\mathsf{diam} \circ \pi^{-1} \circ D_\rho$ is upper semi-continuous. The same logic yields the upper semi-continuity of the map $\mathsf{diam} \circ \pi^{-1} \circ D_\cdot(y_0)$ where

$$D_\cdot(y_0) : \mathbb{R}_0^+ \twoheadrightarrow Y, \ \eta \longmapsto D_\eta(y_0).$$

Thus $\{\eta \geqslant 0 \mid \mathsf{diam}\left(\pi^{-1}\left(D_\eta(y_0)\right)\right) < \varepsilon\}$ is open. It further contains 0 as y_0 has exactly one preimage. Thus we know that there is $\eta_1 > 0$ such that

$$\mathsf{diam}\left(\pi^{-1}\left(D_{\eta_1}(y_0)\right)\right) < \varepsilon.$$

Now this yields that the set

$$A := \left\{y \in Y \;\middle|\; \mathsf{diam}\left(\pi^{-1}\left(D_{\eta_1}(y)\right)\right) < \varepsilon\right\}$$

contains y_0. A is open as $\mathsf{diam} \circ \pi^{-1} \circ D_{\eta_1}$ is upper semi-continuous. Thus there is $\eta_2 > 0$ such that for all $y \in B_{\eta_2}(y_0)$ we have that $\mathsf{diam}\left(\pi^{-1}\left(D_{\eta_1}(y)\right)\right) < \varepsilon$. By monotonicity we learn that for $\eta := \frac{\min(\eta_1, \eta_2)}{2}$ we have $\mathsf{diam}\left(\pi^{-1}\left(B_\eta(y)\right)\right) < \varepsilon$ for any $y \in B_\eta(y)$. $\square$

3.2 Homeomorphy

If the factor map π is a bijection it is an homeomorphism by the

Proposition 3.2.1 (Homeomorphism Criterion)**.** *Let X be a compact and Y be a* Hausdorff *space. Any continuous bijection $f : X \to Y$ is an homeomorphism.*

Proof. Let $A \subseteq X$ be closed. Then A is compact by compactness of X. By continuity $f(A)$ is also compact. As Y is Hausdorff $f(A)$ is closed. Therefore, the image of closed sets is closed. This yields the continuity of f^{-1}. $\square$

Now assume that $\pi : \mathbf{X} \to \mathbf{Y}$ is a homeomorphism, i.e. a conjugacy of dynamical systems. Conjugacy is the notion of isomorphy of `tds`. A property is considered *"dynamical"* if it is preserved by conjugacies. So by definition every *"dynamical"* property of $\mathbf{Y}$ also holds for $\mathbf{X}$ and vice versa. Homeomorphy is the strongest invertibility property and by the Homeomorphism Criterion 3.2.1 it is equivalent to π being invertible on the whole space. We will now consider some ways in which π can be invertible on *"big"* subspaces.

3.3 Almost surely 1-1

Suppose that $\mathbf{Y}$ is uniquely ergodic with ν being the unique invariant probability.

Definition 3.3.1 (Almost surely 1-1)**.** $\pi : \mathbf{X} \to \mathbf{Y}$ is called **almost surely 1-1** if and only if there is a measurable set

$$\hat{B} \subseteq \left\{ y \in Y \;\middle|\; \mathsf{card}\left(\pi^{-1}(\{y\})\right) = 1 \right\}$$

such that $\nu\big(\hat{B}\big) = 1$.

Remark 3.3.1.1. π is almost surely 1-1 if and only if ν-almost all points in Y have exactly one preimage.

Proposition 3.3.2. *Let $X \in \mathsf{Top}$. Suppose that μ is a probability on $\mathfrak{B}X$ with full support. Then any measurable $A \subseteq X$ of full measure, i.e. $\mu(A) = 1$, is dense.*

Proof. As μ has full support any open set has positive measure. Thus a set of full measure can not have any open set in its complement. Hence it must be dense. □

Definition 3.3.3 (Injectivity Points)**.** We call $x \in X$ an **injectivity point** if and only if

$$\forall x' \in X : \pi(x') = \pi(x) \Longrightarrow x' = x \,.$$

Remark 3.3.3.1. x is an injectivity point if and only if x is the only point with the image $\pi(x)$.

Proposition 3.3.4. *Let X be metrizable and d a metric on X. Then*

$$B := \left\{ y \in Y \;\middle|\; \mathsf{card}\left(\pi^{-1}(\{y\})\right) = 1 \right\}$$

is G_δ and hence Borel *measurable.*

Sketch of proof. Observe that

$$B = \bigcap_{n \in \mathbb{N}} \pi_* \mathsf{diam}^{-1}\left(\left[0, \frac{1}{n}\right)\right) \,.$$

$\pi_*\mathsf{diam}$ is upper semi-continuous by Lemma 3.1.2. So $\pi_*\mathsf{diam}^{-1}\left([0, \frac{1}{n})\right)$ is open. □

There is a much stronger measurability result for sets of points with fixed number of preimages:

Proposition 3.3.5. *Let X and Y be Polish spaces and $f : X \to Y$ any continuous map. Then*

$$f_*\mathsf{card} : \mathfrak{B}Y \to \mathbb{N} \cup \{\infty\}\,,\; y \longmapsto \begin{cases} \mathsf{card}(f^{-1}(\{y\})) & \mathsf{card}(f^{-1}(\{y\})) \text{ is finite.} \\ \infty & \text{else.} \end{cases}$$

is universally measurable.

We follow closely the proof presented within [Yoo, 2016, Proof of Lemma 3.1].

Sketch of proof. It suffices to show that for any $k \in \mathbb{N}$ the set

$$S_k := (f_*\mathsf{card})^{-1}(\{i \mid i \geqslant k\})\,.$$

is universally measurable. Consider the Polish space $X^k \times Y$ and the projection

$$\pi_Y : X^k \times Y \to Y, (x_1, \ldots, x_k, y) \mapsto y\,.$$

Note that

$$D_k := \left\{(x_1, \ldots, x_k, y) \in X^k \times Y \;\middle|\; \forall i \in \{1, \ldots, k\} : f(x_i) = y \text{ and } \forall i \neq j : x_i \neq x_j\right\}$$

is Borel measurable. Clearly, $S_k = \pi_Y(D_k)$. Thus S_k is analytic and hence universally measurable by Theorem 1.8.15. □

Now let $\mathbf{X}$ also be uniquely ergodic and let μ denote the unique invariant probability of $\mathbf{X}$.

Lemma 3.3.6. *π is almost surely 1-1 if and only if μ-almost all $x \in X$ are injectivity points.*

Proof. Let $A \subseteq X$ denote the set of injectivity points and $B \subseteq Y$ the set of all points with exactly one preimage. We show $A = \pi^{-1}(B)$. If x is an injectivity point then $y = \pi(x)$ has exactly one preimage. Thus $A \subseteq \pi^{-1}(B)$. Conversely, note that the unique preimage of a point $y \in Y$ such that $\mathsf{card}(\pi^{-1}\{y\}) = 1$ is an injectivity point. Thus $\pi^{-1}(B) \subseteq A$. In the metric case we conclude that both sets are measurable. As $\pi_*\mu = \nu$ this yields the equivalence.

If the spaces are not metrizable A and B are in the completion of the σ-algebras: Assume there is a measurable $\hat{B} \subseteq B$ with $\nu(\hat{B}) = 1$. As $\pi_*\mu = \nu$ we learn that $\hat{A} := \pi^{-1}(\hat{B}) \subseteq A$ has full measure. Thus μ-almost all points are injectivity points.

Conversely assume that μ-almost all points are injectivity points. Then there is an $\hat{A} \subseteq A$ such that $\mu(A) = 1$. The restricted function $\pi|_A : A \to B$ is a bijection. Thus $\pi^{-1}(\hat{B}) = \hat{A}$, where $\hat{B} := \pi(\hat{A})$. As $\pi_*\mu = \nu$ we conclude that $\nu(\hat{B}) = 1$. □

The previous step of the hierarchy is stronger than the current one:

Lemma 3.3.7. *A homeomorphism is an almost surely 1-1 map.*

3.4 Topo-Isomorphy

Definition 3.4.1 (The category $\mathsf{ueCHausDyn}(G)$)**.** Let

$$\mathsf{ueCHausDyn}(G) \subset \mathsf{CHausDyn}(G)$$

be the full subcategory of uniquely ergodic topological dynamical systems.

Definition 3.4.2 (Dynamical Borel Functor 1)**.** For $(X, G, \alpha) \in \mathsf{ueCHausDyn}(G)$ denote $\mathbf{X} = (X, G, \alpha)$ and let $\mu_{\mathbf{X}}$ be the unique α-ergodic probability on $\mathfrak{B}X$. Define a functor

$$\begin{aligned} \hat{\mathfrak{B}} : \mathsf{ueCHausDyn}(G) &\longrightarrow \mathsf{ProbDyn}(G), \\ (X, G, \alpha) &\longmapsto (\mathfrak{B}X, \mu_{\mathbf{X}}, G, g \mapsto \mathfrak{B}\alpha(g, \cdot)) \\ (f : \mathbf{X} \to \mathbf{Y}) &\longmapsto (\hat{\mathfrak{B}}f : \hat{\mathfrak{B}}\mathbf{X} \to \hat{\mathfrak{B}}\mathbf{Y}), \end{aligned}$$

where $U_{\mathsf{ProbDyn}(G)}(\hat{\mathfrak{B}}f) = U_{\mathsf{ueCHausDyn}(G)}(f)$.[67] The continuous group action α is translated into the measurable group action $\mathfrak{B}\alpha$ by simply applying $\hat{\mathfrak{B}}$ on $\alpha(g, \cdot)$ for each $g \in G$ separately. Again both actions are the same if interpreted as actions on sets but they are viewed in different categories.

Remark 3.4.2.1. This functor associates each uniquely ergodic topological dynamical system its corresponding ergodic measure preserving dynamical system.

Definition 3.4.3 (Topo-Isomorphy for Unique Ergodicity)**.**
Let $\mathbf{X}, \mathbf{Y} \in \mathsf{ueCHausDyn}(G)$ and $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. We call π a **topo-isomorphism** if and only if

$$\hat{\mathfrak{B}}\pi : \hat{\mathfrak{B}}\mathbf{X} \longrightarrow \hat{\mathfrak{B}}\mathbf{Y}$$

is an isomorphism.

Remark 3.4.3.1. Unfolding the definitions we learn that a factor map

$$\rho : (A, \mathfrak{A}, \alpha, \mu) \to (B, \mathfrak{F}, \beta, \nu)$$

between $(A, \mathfrak{A}, \alpha, \mu), (B, \mathfrak{F}, \beta, \nu) \in \mathsf{ProbDyn}(G)$ is an isomorphism if and only if ρ is an μ-almost-surely defined map with the following properties:

i) ρ is equivariant with respect to α and β.

ii) There is $X_0 \in \mathfrak{A}$ with $\mu(X_0) = 1$ and $Y_0 \in \mathfrak{F}$ with $\nu(Y_0) = 1$ such that $\rho(X_0) \subseteq Y_0$ and $\rho|_{X_0} : X_0 \to Y_0$ is a bijection.

iii) The almost-surely defined inverse ρ^{-1} is measurable.

iv) And ρ is measure preserving, i.e. $\rho_* \mu = \nu$ and $\rho_*^{-1} \nu = \mu$.

[67] See Remark 1.2.4.1 for a discussion of the forgetful functors.

Remark 3.4.3.2. Note that this is a hybrid property connecting measure theoretic aspects with topological ones. Further note that topo-isomorphy is stronger than just asserting that $\hat{\mathfrak{B}}\mathbf{X} \cong \hat{\mathfrak{B}}\mathbf{Y}$ and $\mathbf{X} \cong \mathbf{Y}$ as the isomorphism must be realized by the factor map.

Remark 3.4.3.3. The main difference between topo-isomorphy and almost sure 1-1-ness is that an almost surely 1-1 map must always have injectivity points, while there are topo-isomorphies without any injectivity points.

The idea of extensions of σ-algebras leads to the

Definition 3.4.4 (Weak Isomorphism)**.** Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{F}, \nu)$ be measure spaces. A map $\pi : X \to Y$ is called a **weak isomorphism** if and only if there are strongly approximable extensions $(X, \mathfrak{A}^*, \mu^*)$ and $(Y, \mathfrak{F}^*, \nu^*)$ such that

$$\pi : (X, \mathfrak{A}^*, \mu^*) \to (Y, \mathfrak{F}^*, \nu^*)$$

is an isomorphism.

Remark 3.4.4.1. This is called *"weak"* as any isomorphism is a weak isomorphism by the trivial extension. On the other hand the sets on which π is invertible need not be measurable with respect to the smaller σ-algebras.

A trivial example of a weak isomorphism which is no isomorphism is the following

Example 3.4.4.1. Consider the measurable spaces

$$X = \Big(\{1, 2, 3, 4\}, \Big\{\emptyset, \{1, 2, 3, 4\}\Big\}\Big)$$

and

$$Y = \Big(\{1, 2\}, \Big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\Big\}\Big)$$

Let μ be the unique probability measure on X. Extend $\nu(\{1\}) = 1$ uniquely to a probability measure ν on Y. Then $\pi : 1 \mapsto 1, 2, 3, 4 \mapsto 2$ is not an isomorphism as it is not even almost surely measurable. However it is a weak isomorphism as we can extend μ to a probability measure $\hat{\mu}$ defined on the whole power-set by $\hat{\mu}(\{1\}) = 1$. Thus π is a weak isomorphism.

The previous step of the hierarchy is stronger than the current one:

Lemma 3.4.5. *An almost surely 1-1 factor map is a topo-isomorphism.*

Now let us consider the non-uniquely ergodic case. We first extend the definition of the functor $\hat{\mathfrak{B}}$.

Definition 3.4.6 (Topological Dynamical Systems with fixed Probability)**.** Let us denote by $\mathsf{measCHausDyn}(G)$ the category of topological dynamical systems equipped with an invariant probability measure. The morphisms are the measure preserving factor maps.

Definition 3.4.7 (Dynamical Borel Functor 2)**.** Define a functor

$$\hat{\mathfrak{B}} : \mathsf{measCHausDyn}(G) \longrightarrow \mathsf{ProbDyn}(G)$$

such that

$$\mathbf{X} := (X, G, \alpha, \mu) \longmapsto (\mathfrak{B}X, G, \mathfrak{B}\alpha, \mu),$$

where a morphism $\pi : \mathbf{X} \to \mathbf{Y}$ of the category $\mathsf{measCHausDyn}(G)$ is send to the unique morphism

$$\hat{\mathfrak{B}}\pi : \mathfrak{B}\mathbf{X} \longrightarrow \mathfrak{B}\mathbf{Y}$$

in the category $\mathsf{ProbDyn}(G)$ such that $U_{\mathsf{measCHausDyn}(G)}(\pi) = U_{\mathsf{ProbDyn}(G)}(\hat{B}\pi)$.[68]

The following definition of topo-isomorphy for non-uniquely ergodic systems can be found in [Fuhrmann et al., 2018, p.81] for the case of compact metric phase spaces.

Definition 3.4.8 (Topo-Isomorphy)**.** Let $\mathbf{X}$ and $\mathbf{Y}$ be topological dynamical systems and $\pi : \mathbf{X} \to \mathbf{Y}$ a topological factor map. We call π a **topo-isomorphy** if and only if for any regular invariant probability μ the measure theoretic factor map

$$\hat{\mathfrak{B}}\pi : \hat{\mathfrak{B}}(\mathbf{X}, \mu) \longrightarrow \hat{\mathfrak{B}}(\mathbf{Y}, \pi_*\mu)$$

is an isomorphism. If $\hat{\mathfrak{B}}\pi : \hat{\mathfrak{B}}(\mathbf{X}, \mu) \to \hat{\mathfrak{B}}(\mathbf{Y}, \pi_*\mu)$ is a weak isomorphism for any regular invariant probability μ we call π a **weak-topo-isomorphy**.

Remark 3.4.8.1. In the case of compact metric spaces any probability is regular by the Theorem of Ulam 1.8.9.

3.5 Almost 1-1

We consider residual sets as a topological analogue of sets of full measure. Note however, that quite often there is a canonical probability on a space which assigns measure zero to some residual sets.

Definition 3.5.1 (Generic)**.** If a property holds for a residual subset we call this property **generic**.

Definition 3.5.2 (Almost 1-1, Generically 1-1)**.** Let $\mathbf{X}, \mathbf{Y} \in \mathsf{CHausDyn}(G)$ and $\pi : \mathbf{X} \to \mathbf{Y}$ be a factor map. We call the factor map π **almost 1-1** (or **generically 1-1**) if and only if the set of injectivity points is residual. The name *"almost 1-1"* is established in the literature, so we will stick with it. However in situations where *"almost 1-1"* is meant but could be easily confused with *"almost surely 1-1"*, we will add *"(generically 1-1)"* for clarity.

Proposition 3.5.3. *Let* $\mathbf{X}$ *be minimal. Suppose that*

$$B := \left\{ y \in Y \;\middle|\; \mathsf{card}\left(\pi^{-1}(\{y\})\right) = 1 \right\}$$

is residual. Then

$$A := \{ x \in X \mid x \textit{ is an injectivity point} \}$$

is residual, i.e. π *is almost 1-1.*

[68]See Remark 1.2.4.1 for a discussion of the forgetful functors.

Proof. Recall that $\pi^{-1}(B) = A$ and $\pi(A) = B$. Let $B \supseteq \bigcap_{n\in\mathbb{N}} O_n$ with O_n open and dense. By continuity $\pi^{-1}(O_n)$ is open. We show that it is also dense. Let $U \subseteq X$ be open. By Lemma 3.1.3 the image $\pi(U)$ contains an open set, and thus must contain a point from O_n. Thus, U contains a point from $\pi^{-1}(O_n)$. As A contains a countable intersection of open and dense sets, A is residual. □

Proposition 3.5.4. *Suppose that X is metrizable with metric d and $\mathbf{X} = (X, G, \alpha)$ is minimal. If π has at least one injectivity point, then π is almost 1-1.*

Proof. Let A denote the set of injectivity points. Let $x \in A$. First, we show that $\alpha(g, x) \in A$ for any $g \in G$. If there is $x' \in X$ such that $\pi(x') = \pi(\alpha(g, x))$, then

$$\begin{aligned}\pi(\alpha(g^{-1}, x')) =& \beta(g^{-1}, \pi(x')) \\ &= \beta(g^{-1}, \pi(\alpha(g, x))) \\ &= \pi(\alpha(g^{-1}, \alpha(g, x))) \\ &= \pi(x)\,.\end{aligned}$$

Thus $\alpha(g^{-1}, x') = x$. Applying $\alpha(g, \cdot)$ we learn that $x' = \alpha(g, x)$. This shows that $\alpha(g, x)$ is an injectivity point. In total $\alpha(G, x) \subseteq A$. By minimality A is dense.

Define $A_n := \pi^{-1}\left(\left\{y \in Y \mid \pi_*\mathsf{diam}\,(y) < \frac{1}{n}\right\}\right)$. A_n is open by Lemma 3.1.2 and the continuity of π. As $A \subseteq A_n$ we further learn that A_n is dense. Observe that $A = \bigcap_{n\in\mathbb{N}} A_n$ and conclude that A is residual. □

Lemma 3.5.5. *Let $X \neq \emptyset$ be compact metric. Suppose that G is σ-compact. Let $\mathbf{X} = (X, G, \alpha)$ and $\mathbf{Y} = (Y, G, \beta)$ be two minimal dynamical systems on compact metric spaces (X, d_X) and (Y, d_Y). A topological factor map $\pi : \mathbf{X} \to \mathbf{Y}$ is subject to the following dichotomy:*[69]

$$\textbf{\textit{either}} \quad \pi \textit{ is almost 1-1} \quad \textbf{\textit{or}} \quad \inf_{y\in Y} \pi_*\mathsf{diam}(y) > 0\,.$$

Proof. If π is almost 1-1 then there is at least one $y \in Y$ with only one preimage, so $\inf_{y\in Y} \pi_*\mathsf{diam} = 0$. Hence, both cases of the dichotomy are disjoint.

Assume that $\inf_{y\in Y} \pi_*\mathsf{diam}(y) = 0$. Pick an increasing sequence of compact subsets $K_n \subseteq G$ such that $G = \bigcup_{n\in\mathbb{N}} K_n$. By compactness of K_n we have

$$\forall\varepsilon > 0 : \exists\delta > 0 : \forall k \in K : d_X(x, x') < \delta \Longrightarrow d_X(\alpha(k, x), \alpha(k, x')) < \varepsilon$$

for any $n \in \mathbb{N}$. Using that construct a decreasing sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \delta_n = 0$ such that

$$d_X(x, x') < \delta_{n+1} \Longrightarrow d_X\left(\alpha(k, x), \alpha(k, y)\right) < \delta_n \tag{14}$$

for all $n \in \mathbb{N}$ and $x, x' \in X$ as well as all $k \in K_n$. In detail we let $\delta_1 = 1$. If δ_n is already defined, choose $\delta_{n+1} < \min(2^{n+1}, \delta_n)$ such that (14) holds for any $k \in K_n$ and $x, x' \in X$. Consider the level sets $A_n := \{y \in Y \mid \pi_*\mathsf{diam}(y) < \delta_n\}$. See that

$$\pi^{-1}\left(\{\beta(k, y)\}\right) = \alpha\left(k, \pi^{-1}\left(\{y\}\right)\right)\,.$$

[69] This is a generalization of Lemma 2.4 in [Downarowicz and Glasner, 2016] also following the proof given there.

Thus $\beta(k,\cdot)$ maps A_{n+1} to A_n for $k \in K_n$. $\pi_*\mathsf{diam}$ is upper semi-continuous. So the level sets A_n are open. Recall that we define

$$\text{denseness}(A) = \inf\left\{\varepsilon > 0 \mid B_\varepsilon(A) = X\right\} .$$

By Proposition 1.5.12 we know that

$$f_n : X \longrightarrow \mathbb{R},\ x \longmapsto \text{denseness}(\alpha(K_n, x))$$

is continuous. The sequence f_n is obviously decreasing and as $\mathbf{X}$ is minimal it converges pointwise to 0 by Proposition 1.5.13. By Dini's Theorem 1.5.3, we learn that the convergence is in fact uniform.

Now let $\varepsilon > 0$. Fix $m \in \mathbb{N}$ such that $f_m < \varepsilon$ uniformly. The assumption that $\pi_*\mathsf{diam}$ has no positive infimum implies that the A_n are non-empty. As the level sets are mapped into each other we can pick $x \in A_{n+m}$ and conclude that $\alpha(K_m, x) \subseteq A_n$. So A_n contains an ε-dense subset. As ε was arbitrary we learn that A_n is dense.

The intersection $A := \bigcap_{n\in\mathbb{N}} A_n$ residual. For any $x \in A$ we have that $\pi_*\mathsf{diam}(x) = 0$ which means that x is an injectivity point. So π is almost 1-1. □

Part II
The Hierarchy

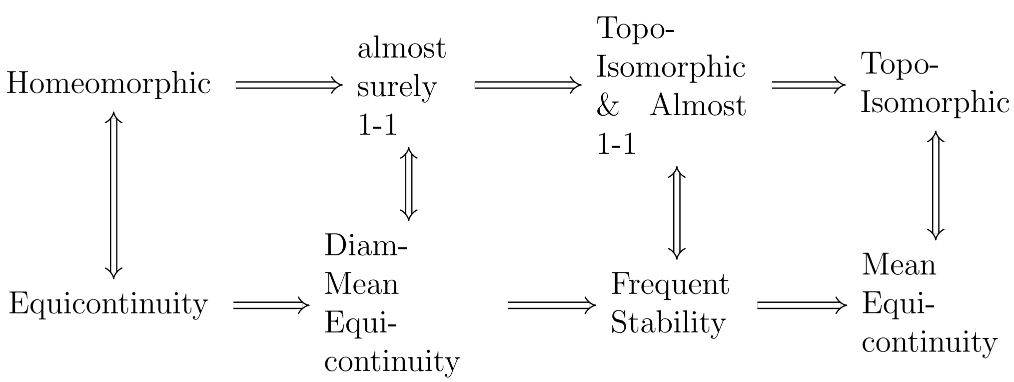

Figure 10: Hierarchy of Dynamical Systems

4 Equicontinuity - Homeomorphic MEF

For this section we assume that G is a topological group. Fix any $(X, G, \alpha) \in \mathsf{CHausDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$.

We will follow the presentation of Joseph Auslander in [Auslander, 1988]. Note that he flips the *"side"* from which G acts on X, i.e. his group actions are anti-actions in our sense. He for example denotes xg for $\alpha(g, x)$ and thus writes xgh for $\alpha(h, \alpha(g, x)) = \alpha(hg, x)$. This inverses the group operation. Using the machinery of `op` developed in the Definition 1.10.3, we can say that a dynamical system of G in his sense is a dynamical system of G^{op} in our sense. Note that he does not consider the opposite `tds` as defined in 1.11.5. We thus need to be cautious and to flip the order of multiplication in his definitions.

Recall the Definition 1.5.1 of equicontinuity. We are going to characterize and explore equicontinuity from various viewpoints.

4.1 Minimal Subsets and Almost Periodicity

Lemma 1.11.9 states that every system with compact phase space has at least one minimal subsystem. However, usually one can not decompose a compact system into minimal subsystems. For example a transitive systems is not necessarily minimal but can not be decomposed into two closed invariant subsystems. We will see that the situation for equicontinuous systems is different.

Definition 4.1.1 (Pointwise Almost Periodicity)**.** Let $\Delta \subseteq X \times X$ be the diagonal. A point $x \in X$ is called **(α)-almost periodic**, if and only if for any neighbourhood $U \in \mathcal{U}(x)$ there is a right syndetic subset $A \subseteq G$ such that

$$\alpha(A, x) \subseteq U\,.$$

If every point $x \in X$ is almost periodic, we call **X** pointwise almost periodic.

Lemma 4.1.2. *Equicontinuity implies pointwise almost periodicity.*[70]

Proposition 4.1.3. *A point is almost periodic if and only if its orbit closure is minimal.*[71]

Remark 4.1.3.1. Note that the minimality of the orbit closure is totally independent of the topology on G. So all the seemingly different notions of almost periodicity with respect to different group topologies, which make the action α continuous, are equivalent. In particular we can equip G with the discrete topology without changing which systems are almost periodic or not.

This independence result is stressed by Auslander in several occasions, e.g. on page 3 of [Auslander, 1988].

Corollary 4.1.3.1. *A point is almost periodic if and only if for any $U \in \mathcal{U}(x)$ there is a* ***finite*** *set $F \subseteq G$ such that $G = F \cdot A$, where $A := \{g \in G : \alpha(g, x) \in U\}$.*[72]

Proposition 4.1.4. *Let G be locally compact. Fix a left* Haar *measure m on G as well as a left ergodic sequence $\mathcal{F}$. If x is almost periodic and $U \in \mathcal{U}(x)$, then the set of return times $A := \{g \in G : \alpha(g, x) \in U\}$ has positive upper density, i.e. $\bar{D}_{\mathcal{F}}(A) > 0$.*

[70] This result can be found as Lemma 3 of Chapter 2 in [Auslander, 1988, p.37].
[71] This is Theorem 7 of Chapter 1 in [Auslander, 1988, p.11].
[72] Cf. Corollary 9 of Chapter 1 [Auslander, 1988, p.12].

Proof. By Corollary 4.1.3.1 there is a finite set $F = \{f_1, \ldots, f_n\}$ such that

$$G = F \cdot A = \bigcup_{i=1}^{n} f_i \cdot A\,.$$

Now calculate

$$\begin{aligned} 1 = \bar{D}_{\mathcal{F}}(G) &= \bar{D}_{\mathcal{F}}\left(\bigcup_{i=1}^{n} f_i \cdot A\right) \\ &\leqslant \sum_{i=1}^{n} \bar{D}_{\mathcal{F}}\left(f_i \cdot A\right) = n \cdot \bar{D}_{\mathcal{F}}\left(A\right)\,. \end{aligned} \tag{15}$$

Where in (15) we use 1.10.25 and Lemma 1.10.24. In total $\bar{D}_{\mathcal{F}}(A) \geqslant \frac{1}{n} > 0$. □

Combining Lemma 4.1.2 and Proposition 4.1.3 we obtain the well-known

Theorem 4.1.5. *Equicontinuous systems can be decomposed into a (necessarily disjoint) union of minimal subsystems.*

The converse is false as there are minimal systems which are not equicontinuous. So Theorem 4.1.5 can not be extended into a characterization of equicontinuity. In fact, it is a direct corollary of Proposition 4.1.3 that a system is decomposable into minimal subsystems if and only if it is pointwise almost periodic.[73] So in order to find an equivalent characterization of equicontinuity we need to use a stricter requirement than pointwise almost periodicity.

Definition 4.1.6 (Uniform Almost Periodicity)**.** The system $\mathbf{X}$ is uniformly almost periodic, if and only if for any neighbourhood ε of the diagonal in $X \times X$ there is a right syndetic subset $A \subseteq G$ such that

$$\forall x \in X : \forall a \in A : (\alpha(a, x), x) \in \varepsilon\,.$$

The difference to pointwise almost periodicity is that the syndetic subset must be chosen uniformly for all points.

Theorem 4.1.7 (Characterization via Uniform Almost Periodicity)**.** *A system is equicontinuous if and only if it is uniformly almost periodic.*[74]

4.2 Automorphism Group

Observe that for any `tds` $\mathbf{Z} = (Z, G, \gamma)$ its automorphism group is given by

$$\mathrm{Aut}(\mathbf{Z}) = \{f \in \mathrm{Homeo}(Z) \mid \forall g \in G : \gamma(g, \cdot) \circ f = f \circ \gamma(g, \cdot)\}\,.$$

Conclude that $\mathrm{Aut}(\mathbf{Z})$ is exactly the group theoretic centralizer of $\gamma(G, \cdot)$ in the group of all homeomorphisms $\mathrm{Homeo}(X)$. We understand automorphisms as symmetries of the system.

[73] See [Auslander, 1988, Chapter 1, Corollary 10].
[74] This is Theorem 2 of Chapter 2 in [Auslander, 1988, p.36].

Definition 4.2.1 (Homogeneous `tds`)**.** The system $\mathbf{X}$ will be called **homogeneous** if and only if the automorphism group acts algebraically transitively, i.e.

$$\forall x, y \in X : \exists s \in \operatorname{Aut}(\mathbf{X}) : s(x) = y\,.$$

The topology of uniform convergence is a group topology on $\operatorname{Aut}(\mathbf{X})$. We will write $\operatorname{Aut}_{\mathsf{Top}}(\mathbf{X})$ for the automorphism group as a topological group.

Theorem 4.2.2. *Let the acting group G be Abelian. If $\mathbf{X}$ is minimal and equicontinuous, then $\mathbf{X}$ is homogeneous.*

Proof. Let $z, y \in X$ be arbitrary. By minimality y is in the orbit closure of z. Thus there is a net $\mathbf{g} \in G^I$, where $(I, <)$ is a directed set, such that $\lim_{i\in I} \alpha(\mathbf{g}_i, z) = y$. By compactness of X the family

$$\mathcal{F} := \{x \mapsto \alpha(\mathbf{g}_i, x) \mid i \in I\}$$

is fiber-wise relatively compact. As $\mathcal{F}$ is a subset of the whole equicontinuous family of all transformations induced by G, it is also equicontinuous. Thus by Arzelà-Ascoli Theorem 1.5.2 the family $\mathcal{F}$ is relatively compact in the topology of uniform convergence. In particular there is a uniformly convergent subnet indexed by a directed set K, i.e.

$$\alpha(\mathbf{g}_{\mathbf{i_k}}, \cdot) \xrightarrow[\text{unif.}]{k\in K} g_y\,.$$

Clearly $g_y(z) = y$. Note that by Abelianity we have $\mathcal{F} \subseteq \operatorname{Aut}(\mathbf{X})$. As $\operatorname{Aut}(\mathbf{X})$ is closed upon uniform convergence we conclude $g_y \in \operatorname{Aut}(\mathbf{X})$. Now let $x \in X$ be arbitrary. Then $g_x^{-1} g_y(x) = y$ and $g_x^{-1} g_y \in \operatorname{Aut}(\mathbf{X})$. □

In [Auslander, 1988, Chapter 2, Theorem 13] one can find a converse to that theorem, this time we can get rid of the Abelianity assumption. However we need to add the assumption that the phase space is metrizable.

Theorem 4.2.3. *Let the phase space X be a compact metric space. If $\mathbf{X}$ is homogeneous and minimal, then $\mathbf{X}$ is equicontinuous.*

Corollary 4.2.3.1. *Let $A \leq \operatorname{Homeo}(X)$ be an Abelian subgroup of the group of Homeomorphisms such that the natural action of A on X is algebraically transitive. Then any subgroup of $G \leq A$ is equicontinuous.*

Proof. Note that by Abelianity A is a subgroup of the automorphism group of the natural action of G on X. Thus the automorphism group acts algebraically transitively. □

Combining the Theorems 4.2.2 and 4.2.3 we obtain that

Theorem 4.2.4 (Characterization via Homogeneity)**.** *Let the phase space X be compact metric and the acting group G be Abelian. Suppose that $\mathbf{X}$ is minimal. The system $\mathbf{X}$ is equicontinuous if and only if $\mathbf{X}$ is homogeneous.*

This is one of the many ways in which equicontinuous systems are *simple*. In equicontinuous systems every point behaves exactly the same as any other point under the dynamics as there is a symmetry exchanging the two.

4.3 Enveloping Semigroup

This section follows closely the presentation given in [Auslander, 1988, Chapter 2].

Equip the set X^X of all functions $X \to X$ with the product topology. By TYCHONOFF's Theorem this yields a compact HAUSDORFF space. Further composition of maps $\circ$ yields a monoid $(X^X, \circ)$. As left multiplication is not necessarily continuous in the product topology this is usually not a topological monoid.

Definition 4.3.1 (Enveloping Semigroup)**.** The system $\mathbf{X}$ corresponds to a subgroup of X^X via

$$\alpha(G, \cdot) = \{\alpha(g, \cdot) \mid g \in G\} \leq X^X .$$

We then define

$$E(\mathbf{X}) := \mathrm{cl}(\alpha(G, \cdot))$$

and call it the **enveloping semigroup**.

Lemma 4.3.2. *As a topological space* $E(\mathbf{X})$ *is compact* HAUSDORFF*.*

Proof. X^X is compact HAUSDORFF and $E(\mathbf{X})$ is a closed subspace. □

Proposition 4.3.3. *The enveloping semigroup* $E(\mathbf{X})$ *is a monoid (and indeed a semigroup).*

Proof. Clearly $\alpha(1_G, \cdot) \in \alpha(G, \cdot) \leq E(\mathbf{X})$ is the identity.

Note that for any $g \in G$ the mapping

$$\lambda_g : X^X \longrightarrow X^X,\, f \longmapsto \alpha(g, \cdot) \circ f$$

is continuous. Of course, $\lambda_g(\alpha(G, \cdot)) \subseteq \alpha(G, \cdot)$. Now calculate

$$\lambda_g(E(\mathbf{X})) = \lambda_g(\mathrm{cl}(\alpha(G, \cdot))) \subseteq \mathrm{cl}\left(\lambda_g(\alpha(G, \cdot))\right) \subseteq \mathrm{cl}(\alpha(G, \cdot)) = E(\mathbf{X}) .$$

We conclude that

$$\alpha(G, \cdot) \cdot E(\mathbf{X}) \subseteq E(\mathbf{X}) . \tag{16}$$

Note that for $e \in E(\mathbf{X})$ we further have that

$$\rho_e : X^X \longrightarrow X^X,\ f \longmapsto f \circ e$$

is continuous. By Equation (16) we have $\rho_e(\alpha(G, \cdot)) \subseteq \alpha(G, \cdot)$ for $e \in E(\mathbf{X})$ By continuity again we see that $\rho_e(E(\mathbf{X})) \subseteq E(\mathbf{X})$. We thus obtain

$$E(\mathbf{X}) \cdot E(\mathbf{X}) \subseteq E(\mathbf{X}) .$$

□

Remark 4.3.3.1. Note that $E(\mathbf{X})$ is usually not a group. This is due to right-multiplication (with non-continuous functions) not being continuous in the topology of pointwise convergence. Thus we can not carry over inverses to the limit.

We can extend the continuous group action α of G to a continuous monoid action $\hat{\alpha}$ of $E(\mathbf{X})$ by $\hat{\alpha}(e, x) = e(x)$.

Proposition 4.3.4. *For any $x \in X$ we have*

$$\hat{\alpha}(E(\mathbf{X}), x) = \mathrm{cl}(\alpha(G, x))\,.$$

This means that the enveloping semigroup allows to replace orbit closures by orbits.[75]

We can also view $\alpha(G, \cdot)$ as a subset of the space $\mathcal{C}(X, X)$ equipped with the topology of uniform convergence. Right multiplication is continuous in the topology of uniform convergence so:

Lemma 4.3.5. *Let $F := \mathrm{cl}_{\mathit{unif.}}(\alpha(G, \cdot)$ be the closure of $\alpha(G, \cdot)$ in the topology of uniform convergence. Then F is a topological group.*

Proof. Clearly $\alpha(G, \cdot)$ is a topological group. We thus must show that the properties of groups are preserved by the closure. As the composition $\circ$ is continuous we see that

$$\begin{aligned}\circ(F, F) &= \circ\Big(\mathrm{cl}_{\text{unif.}}\left(\alpha(G, \cdot)\right), \mathrm{cl}_{\text{unif.}}\left(\alpha(G, \cdot)\right)\Big)\\ &\subseteq \mathrm{cl}_{\text{unif.}}\Big(\circ\left(\alpha(G, \cdot), \alpha(G, \cdot)\right)\Big)\\ &= \mathrm{cl}_{\text{unif.}}(\alpha(G, \cdot)) = F\,.\end{aligned}$$

So F is indeed a semigroup. Inverses exist as for $f \in F$ there is a net $\mathbf{f} \in \alpha(G, \cdot)^I$ for some directed set $(I, <)$ such that

$$\mathbf{f}_i \xrightarrow[\text{unif.}]{i \in I} f\,.$$

Then by continuity of the inverse we have that

$$\mathbf{f}_i^{-1} \xrightarrow[\text{unif.}]{i \in I} g\,.$$

Now clearly

$$\mathrm{Id}_X = \lim_{i \in I} \mathbf{f}_i \circ \mathbf{f}_i^{-1} = f \circ g\,.$$

Thus F is indeed a group. □

Lemma 4.3.5 combined with the Arzelà-Ascoli Theorem 1.5.2 shows that if $\mathbf{X} = (X, G, \alpha)$ is equicontinuous then there is a compact group F extending the action. As actions of compact groups are always equicontinuous and equicontinuity is preserved by restricting to subgroups we obtain yet another characterization of equicontinuity.

Theorem 4.3.6 (Characterization via Group Extensions)**.** $\mathbf{X}$ *is equicontinuous if and only if $\alpha(G, \cdot)$ admits a compact group extension in $\mathcal{C}(X, X)$ with respect to the topology of uniform convergence.*[76]

[75] This is Proposition 1 of Chapter 3 in [Auslander, 1988, p.51].

[76] [Auslander, 1988, Chapter 3, Remark after Theorem 2]

Now for equicontinuous systems we see that the enveloping semigroup is in fact a group and coincides with the closure of $\alpha(G,\cdot)$ in the topology of uniform convergence.

Theorem 4.3.7 (Characterization via Enveloping Semigroup)**.** *The system* $\mathbf{X}$ *is equicontinuous if and only if its enveloping semigroup* $E(\mathbf{X})$ *is a group of homeomorphisms.*[77]

Proof. Suppose that $E(\mathbf{X})$ is indeed a group of homeomorphisms. By TYCHONOFF's Theorem we know that $E(\mathbf{X})$ is compact. Thus $E(\mathbf{X}) \subseteq \mathcal{C}(X,X)$ is a compact group extension of $\alpha(G,\cdot)$. By Theorem 4.3.6 we conclude that $\mathbf{X}$ is equicontinuous.

Conversely, let $\mathbf{X}$ be equicontinuous. Equip $\mathcal{C}(X,X)$ with the topology of uniform convergence and X^X with the topology of pointwise convergence . Let

$$\iota : \mathcal{C}(X,X) \longhookrightarrow X^X,\ f \longmapsto f$$

be the inclusion of the sets $\mathcal{C}(X,X) \subseteq X^X$. Uniformly converging nets also converge pointwise. Thus, ι is continuous.

In order to avoid confusion let us introduce notation: Let cl_u denote closure w.r.t. uniform convergence and cl_p closure w.r.t. pointwise convergence. As $\mathbf{X}$ is equicontinuous, ARZELÀ-ASCOLI Theorem 1.5.2 tells us that $\mathrm{cl}_u(\alpha(G,\cdot))$ is compact. As ι is continuous $\iota(\mathrm{cl}_u(\alpha(G,\cdot)))$ is compact. Now X^X is HAUSDORFF and thus $\iota(\mathrm{cl}_u(\alpha(G,\cdot)))$ is closed. Therefore, $\iota\left(\mathrm{cl}_u(\alpha(G,\cdot)\right) \supseteq \mathrm{cl}_p(\alpha(G,\cdot))$. Conversely, for any continuous map φ we have $\varphi(\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(\varphi(A))$, as $\varphi^{-1}\left(\mathrm{cl}_Y(\varphi(A))\right)$ is a closed superset of A. This yields in our situation that $\iota\left(\mathrm{cl}_u(\alpha(G,\cdot)\right) \subseteq \mathrm{cl}_p(\alpha(G,\cdot))$. Thus in total $\iota\left(\mathrm{cl}_u(\alpha(G,\cdot))\right) = \mathrm{cl}_p(\alpha(G,\cdot))$.

Denoting $F := \mathrm{cl}_u(\alpha(G,\cdot)$ we conclude that

$$\iota|_F : \mathrm{cl}_u(\alpha(G,\cdot) \longrightarrow \mathrm{cl}_p(\alpha(G,\cdot))$$

is a bijection. By the HOMEOMORPHISM CRITERION $\iota|_F$ is thus an homeomorphism. This shows that in F pointwise and uniform convergence are equivalent. By Lemma 4.3.5 we obtain that $F = E(\mathbf{X})$ is a group. ☐

AUSLANDER shows in [Auslander, 1988, Chapter 3, Theorem 7] that

Theorem 4.3.8. *Let* $\mathbf{X}, \mathbf{Y} \in \mathsf{CHausDyn}(G)$ *and* $\pi : \mathbf{X} \to \mathbf{Y}$ *be a factor map. Then there is a unique continuous semigroup homomorphism* $\Phi_\pi : E(\mathbf{X}) \to E(\mathbf{Y})$ *such that* $\pi(p(x)) = (\Phi_\pi(p))\,(\pi(x))$ *for all* $x \in X$ *and* $p \in E(\mathbf{X})$

This provides a potential functor from the category of topological dynamical system in the category of compact left-continuous semigroups $\mathsf{clcSemiGrp}$

$$\begin{aligned} E : \mathsf{CHausDyn}(G) &\longrightarrow \mathsf{clcSemiGrp}\,, \\ \mathbf{X} &\longmapsto E(\mathbf{X})\,, \\ \pi &\longmapsto \Phi_\pi\,. \end{aligned}$$

[77][Auslander, 1988, Chapter 3, Theorem 3]. The proof given here follows closely the proof given there.

Corollary 4.3.8.1. *E is indeed a covariant functor.*

Proof. The only thing remaining to be shown is that $\Phi_{\phi\circ\rho} = \Phi_\phi \circ \Phi_\rho$. Observe that

$$\pi \circ \rho(p(x)) = (\Phi_{\pi\circ\rho}(p))\,(\pi \circ \rho(x))$$

and

$$\begin{aligned}\pi \circ \rho(p(x)) &= \pi\left((\Phi_\rho(p))\,(\rho(x))\right)\\ &= (\Phi_\pi(\Phi_\rho(p)))\,(\pi \circ \rho(x))\ .\end{aligned}$$

Now, uniqueness yields $\Phi_{\pi\circ\rho} = \Phi_\pi \circ \Phi_\rho$. □

4.4 Classification of Minimal Equicontinuous Systems

The following classification theorem can be found in [Auslander, 1988, Chapter 3, Theorem 6]. Vividly speaking, it states that (in the case of an Abelian acting group) any minimal and equicontinuous system is conjugate to the rotation associated to a group compactification as defined in Definition 1.11.16. This can be interpreted category theoretically as functors between the category of **pointed** minimal and equicontinuous systems $\mathsf{MinEquiDyn}_{\mathsf{pt}}(G)$ and the category of group compactifications $\mathsf{Comp}(G)$. In fact Theorem 4.16 in [Hermle and Kreidler, 2022, p.23] states the equivalence of $\mathsf{MinEquiDyn}_{\mathsf{pt}}(G)$ and $\mathsf{Comp}(G)$ whenever G is Abelian.

Theorem 4.4.1. *Let $\mathbf{X} = (X, G, \alpha)$ be equicontinuous and minimal. For any $x \in X$ let*

$$\mathrm{Stab}_x(E(\mathbf{X})) := \{p \in E(\mathbf{X}) \mid p(x) = x\}$$

be the stabilizer of x w.r.t. the action of $E(\mathbf{X})$ on X. Then $\mathrm{Stab}_x := \mathrm{Stab}_x(E(\mathbf{X}))$ is a closed subgroup of $E(\mathbf{X})$. Let $\bar\alpha$ denote the promotion of the action α of G on X to an action on $E(\mathbf{X})$ by

$$\bar\alpha(g, f) = \alpha(g, \cdot) \circ f\ .$$

G acts on the homogeneous space $E(\mathbf{X})/\mathrm{Stab}_x$ by

$$\begin{aligned}\bar\alpha/\mathrm{Stab}_x :\ G \times E(\mathbf{X})/\mathrm{Stab}_x &\longrightarrow E(\mathbf{X})/\mathrm{Stab}_x,\\ (g, f\mathrm{Stab}_x) &\longmapsto \bar\alpha(g, f)\mathrm{Stab}_x\ .\end{aligned}$$

The systems (X, G, α) and $(E(\mathbf{X})/\mathrm{Stab}_x, G, \bar\alpha/Stab_x)$ are conjugate. If G is Abelian, then Stab_x is trivial and (X, G, α) is conjugate to $(E(\mathbf{X}), G, \bar\alpha)$.

Proof. The main idea is the classical orbit-stabilizer theorem from abstract algebra, stating that for any group action the group modulo the stabilizer of one point is bijective to the orbit of that point. Recall the action of $E(\mathbf{X})$ on X given by $\hat\alpha(e, x) = e(x)$. As the $\mathrm{cl}(\alpha(G, x)) = \hat\alpha(E(\mathbf{X}), x)$ we can apply this to our setting. By minimality $\mathrm{cl}(\alpha(G, x)) = X$.

We divide the proof into various steps.

The stabilizer is a subgroup: Let $p \in \mathrm{Stab}_x$, i.e. $p(x) = x$. Then

$$x = p^{-1}(p(x)) = p^{-1}(x) .$$

Thus $p^{-1} \in \mathrm{Stab}_x$. Now let $q \in \mathrm{Stab}_x$. We calculate $p(q(x)) = p(x) = x$. Thus $p \circ q \in \mathrm{Stab}_x$.

The stabilizer is closed: Stab_x is the preimage of $\{x\}$ under the evaluation map $f \mapsto f(x)$. This evaluation map is continuous w.r.t. the product topology on X^X. Thus Stab_x is closed in the product topology on $E(\mathbf{X}) \subseteq X^X$. By the proof of Theorem 4.3.7 the product topology on $E(\mathbf{X})$ coincides with the topology of uniform convergence.

G acts on the cosets: Note that the action α extends to an action $\bar{\alpha}$ on the enveloping semigroup by left composition. Note that

$$f\mathrm{Stab}_x = \{p \in E(\mathbf{X}) \mid p(x) = f(x)\} .$$

Define the action $\bar{\alpha}/\mathrm{Stab}_x$ by

$$\bar{\alpha}/\mathrm{Stab}_x\,(g, f\mathrm{Stab}_x) := \bar{\alpha}(g, f)\mathrm{Stab}_x = \{q \in E(\mathbf{X}) \mid q(x) = \alpha(g, f(x))\} .$$

Pick any other representative $g \in f\mathrm{Stab}_x$. All elements in $f\mathrm{Stab}_x$ are characterized via their value at x and $\{q \in E(\mathbf{X}) \mid q(x) = \alpha(g, f(x))\}$ only depends on the value at x. So we see that $\bar{\alpha}/\mathrm{Stab}_x$ is well-defined.

The systems are conjugate: Let

$$h : E(\mathbf{X})/\mathrm{Stab}_x \longrightarrow X,\ f\mathrm{Stab}_x \longmapsto f(x) .$$

I claim that h is an homeomorphism. Let $g, f \in E(\mathbf{X})$ such that $f(x) = g(x)$. Then $f \in g\mathrm{Stab}_x$ and thus $f\mathrm{Stab}_x = g\mathrm{Stab}_x$. Thus h is injective. Now we show that h is surjective. Clearly $h(E(\mathbf{X})) = \bar{\alpha}(E(\mathbf{X}), x)$. Proposition 4.3.4 yields that $\bar{\alpha}(E(\mathbf{X}), x) = \mathrm{cl}\,(\alpha(G, x)) = X$. Now observe, that h is a projection map and thus continuous by the definition of the product topology. As a continuous bijection between compact Hausdorff spaces h is a homeomorphism.

It remains to show the equivariance. Calculate

$$\begin{aligned} h\left(\bar{\alpha}/\mathrm{Stab}_x(g, f\mathrm{Stab}_x)\right) &= h\left(\bar{\alpha}(g, f)\mathrm{Stab}_x\right) \\ &= \bar{\alpha}(g, f)(x) = \alpha(g, f(x)) \\ &= \alpha\left(g, h(f\mathrm{Stab}_x)\right) . \end{aligned}$$

Triviality of stabilizer in Abelian case Note that as G is Abelian also $E(\mathbf{X})$ is Abelian. Thus any $p \in \mathrm{Stab}_x$ is an automorphism. By the dichotomy in Lemma 3.1.1 and the fact that $\mathrm{Id}_X(x) = p(x)$ we conclude $\mathrm{Id}_X = p$. □

Remark 4.4.1.1. a) Note that $E(\mathbf{X})/\mathrm{Stab}_x$ is a group if and only if Stab_x is a normal subgroup. On the other hand as $g\mathrm{Stab}_x g^{-1} = \mathrm{Stab}_{\alpha(g,x)}$ we learn that if Stab_x is normal it must be trivial. In order to see this assume that Stab_x is normal. Let $p \in \mathrm{Stab}_x$, then $p \in \mathrm{Stab}_{\alpha(g,x)}$ for any $g \in G$. Thus $p(\alpha(g,x)) = \alpha(g,x)$ for all $g \in G$. By continuity and minimality $p = \mathrm{Id}_X$.

There are however many examples of non-Abelian group actions with trivial stabilizers. Say for example a non-Abelian compact group acting on itself by left-multiplication.

b) The theorem is optimal in the following sense: Suppose the stabilizer is non-trivial, i.e. not normal, then the system is not conjugate to a group rotation.

Proof. Let Q be a topological group and $\eta : G \to Q$ a continuous group homomorphism. Suppose that (X, G, α) is conjugate to (Q, G, β) with

$$\beta(q, g) = \eta(g)q\,.$$

Let

$$\Psi : (Q, G, \beta) \longrightarrow (E(\mathbf{X})/\mathrm{Stab}_x, G, \bar{\alpha}/\mathrm{Stab}_x)$$

denote the induced conjugation. By conjugacy we know that $g \mapsto \bar{\alpha}(g,\cdot)$ must have the same kernel as η. Without restriction we can assume those kernels are trivial. Let $e\mathrm{Stab}_x := \Psi(1_Q)$ be the image of the unit element in Q. By potentially composing Ψ with an automorphism, we can assume that $e\mathrm{Stab}_x = \mathrm{Stab}_x$.

We define a group structure on $E(\mathbf{X})/\mathrm{Stab}_x$ by

$$\bar{\alpha}(g,\cdot)\mathrm{Stab}_x \odot \bar{\alpha}(h,\cdot)\mathrm{Stab}_x := \Psi(\eta(gh))\,.$$

Note that

$$\begin{aligned}
\bar{\alpha}(g,\cdot)\mathrm{Stab}_x \odot \bar{\alpha}(h,\cdot)\mathrm{Stab}_x &= \Psi(\eta(gh)) \\
&= \Psi(\eta(g)\eta(h)) \\
&= \bar{\alpha}/\mathrm{Stab}_x(g, \Psi(\eta(h))) \\
&= \bar{\alpha}(g,\cdot)\Psi(\eta(h)) \\
&= \bar{\alpha}(g,\cdot) \circ \bar{\alpha}(h,\cdot)\mathrm{Stab}_x\,.
\end{aligned}$$

Thus the group operation is the usual multiplication of cosets. By continuity and minimality this extends to the whole space of cosets $E(\mathbf{X})/\mathrm{Stab}_x$. However this operation is not well-defined as Stab_x is not normal. This is a contradiction. □

Corollary 4.4.1.1. *Let G be left (or right) amenable. If $\mathbf{X}$ is minimal and equicontinuous, then $\mathbf{X}$ is uniquely ergodic.*

Proof. By Theorem 1.13.2 there is an $\bar{\alpha}/\mathrm{Stab}_x$-invariant inner regular Borel probability measure μ on $E(\mathbf{X})/\mathrm{Stab}_x$. The action $\bar{\alpha}/\mathrm{Stab}_x$ corresponds to left-multiplication by the dense subgroup $\alpha(G,\cdot) \leq E(\mathbf{X})$. Let $e_*\mu$ denote the probability $e_*\mu(A) = \mu(e^{-1}A)$ on H/G for $e \in E(\mathbf{X})$. Let $\varepsilon > 0$. There is an open set $U \supseteq A$ such that $\mu(U \setminus A) < \varepsilon$. Now $e \mapsto e^{-1}A$ is upper hemi-continuous by Theorem 1.5.9. So $V := \{e \in E(\mathbf{X}) \mid e^{-1}A \subseteq U\}$ is an open neighbourhood of the identity. Clearly $e_*\mu(A) < \mu(A) + \varepsilon$ for $e \in V$. This proves the upper semi-continuity of $e \mapsto e_*\mu(A)$.

So the superlevel set $E' := \{e \in E(\mathbf{X}) \mid e_*\mu(A) \geqslant \mu(A)\}$ is closed for any measurable A. As $\alpha(G, \cdot) \subseteq E'$ is dense we conclude $E' = E(\mathbf{X})$.

Now suppose that there is a measurable A and $e \in E(\mathbf{X})$ such that $e_*\mu(A) > \mu(A)$. Then $e_*^{-1}\mu(eA) < \mu(A)$. A contradiction.

So $e_*\mu = \mu$ for all $e \in E(\mathbf{X})$. This means that μ is invariant by the left-multiplication of $E(\mathbf{X})$ on the homogeneous space $E(\mathbf{X})/\mathrm{Stab}_x$. As $E(\mathbf{X})$ is compact Hausdorff we can apply Theorem 1.10.13. We learn that there is only one probability invariant under left-multiplication. In particular there can be no $\bar{\alpha}/\mathrm{Stab}_x$-invariant probability $\nu \neq \mu$. So $\mathbf{X}$ is uniquely ergodic. □

Theorem 4.4.1 shows that in the Abelian case any minimal equicontinuous system is a group rotation on a compact group by a dense subgroup. We can interpret the proof in a more direct way:

Theorem 4.4.2. *Let G be Abelian. Let $\mathbf{X} = (X, G, \alpha)$ be equicontinuous and minimal. For $x \in X$ there is an Abelian group operation $\oplus$ on X such that $\alpha(G, x)$ is a subgroup and that $\mathbf{X}$ is conjugate to the system (X, G, β) with $\beta(g, y) = \alpha(g, x) \oplus y$.*

Interpretation for Abelian case. First we construct the Abelian group operation. Fix any $x \in X$ and define $0_\mathbf{X} := x$. The idea to define the addition is the following: By minimality and equicontinuity we can find an automorphism R_y such that $R_y(x) = y$. By the dichotomy Lemma 3.1.1 R_y is unique. This is interpreted as the way that x is transformed into y. We will now transport this transformation to any base point z and define $z \oplus y := R_y(z)$. This will turn out to be well-defined as R_y is unique.

Now for the technical part. By minimality there is a net $(g_i)_{i \in I}$ such that

$$\alpha(g_i, x) \xrightarrow[i \in I]{} y\,.$$

By equicontinuity and Arzelà-Ascoli Theorem (1.5.2) there is a convergent sub-net $(g_j)_{j \in J}$ such that

$$\alpha(g_j, \cdot) \xrightarrow[j \in J]{} R_y \in \mathcal{C}(X, X)$$

uniformly. Further there is a sub-net $(g_k)_{k \in K}$ such that

$$\alpha(g_k^{-1}, \cdot) \xrightarrow[k \in K]{} g \in \mathcal{C}(X, X)$$

uniformly. By continuity of the composition of continuous functions we obtain that $g \circ R_y = \mathrm{Id} = R_y \circ g$. Thus R_y is invertible.

Suppose h is another limit point of $\alpha(G, \cdot)$ such that $h(x) = y$. Both are automorphisms, thus the dichotomy in Lemma 3.1.1 yields that $h = R_y$.

We define $z \oplus y := R_y(z)$ as well as $-y := R_y^{-1}(x)$. Note that thus $R_{-y} = R_y^{-1}$ by uniqueness. As composition of functions is associative we obtain that $\oplus$ is associative. We further see

$$y \oplus -y = R_{-y}(y) = R_y^{-1}(y) = R_y^{-1}(R_y(x)) = x = 0_\mathbf{X}\,.$$

Thus we indeed have defined a group structure on $\mathbf{X}$.

Define

$$\Psi : G \to X, g \mapsto \alpha(g, x) \,.$$

This is a group homomorphism. First note that $R_{\alpha(g,x)} = \alpha(g, \cdot)$. Then

$$\begin{aligned}\Psi(gh) &= \alpha(gh, x) = \alpha(hg) = \alpha(h, \alpha(g, x)) \\ &= R_{\alpha(h,x)}(\alpha(g, x)) = \alpha(g, x) \oplus \alpha(h, x) \\ &= \Psi(g) \oplus \Psi(h) \,.\end{aligned}$$

Further for any $y \in X$ we have

$$\Psi(g) \oplus y = R_y(\alpha(g, x)) = \alpha(g, R_y(x)) = \alpha(g, y) \,.$$

Thus the dynamics is indeed given by group rotations. This shows further that with the original topology on X we obtain a topological group.

Finally observe that $\alpha(G, x)$ is dense in X by minimality. Thus Ψ has dense image and further $(X, \oplus)$ is Abelian. □

Recall the above Definition 1.11.16 of the functor $\mathsf{Rot} : \mathsf{Comp}(G) \to \mathsf{EquiDyn}_{\mathsf{pt}}(G)$. Clearly, for any $K \in \mathsf{Comp}(G)$ the `tds` $\mathsf{Rot}(K)$ is minimal (and equicontinuous), thus we obtain

Theorem 4.4.3 (Characterization via Group Rotations)**.** *Let the acting group G be Abelian. Suppose that $\mathbf{X}$ is minimal. $\mathbf{X}$ is equicontinuous if and only if there is $K \in \mathsf{Comp}(G)$ such that $\mathbf{X}$ is a rotation on K.*

As equicontinuous systems are a disjoint union of minimal equicontinuous systems we obtain

Theorem 4.4.4. *Any equicontinuous system with Abelian acting group is conjugate to a disjoint union of group rotations by dense subgroups.*

There is one further observation to be made.

Proposition 4.4.5. *Let the acting group G be Abelian. For any minimal and equicontinuous system $\mathbf{X}$ the automorphism group and the enveloping semigroup coincide, i.e. $E(\mathbf{X}) = \mathrm{Aut}(\mathbf{X})$.*

Proof. By Abelianity we clearly have that any element of $E(\mathbf{X})$ commutes with any transformation in $\alpha(G, \cdot)$. Thus $E(\mathbf{X}) \subseteq \mathrm{Aut}(\mathbf{X})$. Now let $h \in \mathrm{Aut}(\mathbf{X})$. Define $h(x) =: y$. By minimality and equicontinuity we can find a uniform limit f of a net in $\alpha(G, \cdot)$ such that $f(x) = y$. Now $f(x) = h(x)$ and by the dichotomy Lemma 3.1.1 we conclude $h = f \in E(\mathbf{X})$. □

4.5 Continuous Eigenfunctions

We follow the presentation given in [Hermle and Kreidler, 2022].

Let E be a Banach space. By $\mathfrak{L}(E, E)$ let us denote the space of bounded linear operators $E \to E$.

Definition 4.5.1 (Group Representation)**.** We call a group homomorphism $\Phi : G \to \mathfrak{L}(E,E)$ a **strongly continuous group representation** if and only if for any sequence $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ with limit $g \in G$ we have $\Phi(g_n) \xrightarrow{n\to\infty} \Phi(g)$ pointwise.

There is a canonical strongly continuous group representation $C_{\mathbf{X}}$ associated to a topological dynamical system $\mathbf{X} = (X, G, \alpha)$, called the KOOPMAN representation: As the (complex) Banach space we choose the space $E = \mathcal{C}(X, \mathbb{C})$ with the supremum norm. Then the representation is given by $(C_{\mathbf{X}}(g))\,(f) = f \circ \alpha(g, \cdot)$.

Definition 4.5.2 (Generalized Discrete Spectrum)**.** We say that a bounded and strongly continuous group representation Φ has **discrete spectrum** if and only if

$$E = \bigcup \mathrm{cl}\,\{V \leq E \mid V \text{ is finite dimensional with } \Phi(g)(V) \subseteq V \text{ for } g \in G\}\ .$$

Definition 4.5.3 (dsce)**.** We say that $\mathbf{X}$ has discrete spectrum of continuous eigenfunctions (dsce) if and only if the group representation $C_{\mathbf{X}}$ has discrete spectrum.

Theorem 1.11 in [Hermle and Kreidler, 2022, p.9] states that

Theorem 4.5.4 (Characterization of dsce)**.** $\mathbf{X}$ *has discrete spectrum of continuous eigenfunctions if and only if* $\mathbf{X}$ *is equicontinuous.*

Let G be Abelian.

Definition 4.5.5 (Spectrum of a Representation)**.** Let E be a Banach space and $\Phi : G \to \mathfrak{L}(E,E)$ be a bounded and strongly continuous representation. We call a $\chi \in G^*$ an **eigencharacter** if and only if there is $x \in E \setminus \{0\}$ such that for any $g \in G$ we have

$$(\Phi(g))(x) = \chi(g) \cdot x\ .$$

The **spectrum** of Φ is given by the set of all eigencharacters

$$\sigma_{\mathsf{p}}(\Phi) := \{\chi \in G^* \mid \chi \text{ is eigencharacter of } \Phi\}\ .$$

For $\mathbf{X} \in \mathsf{CHausDyn}(G)$ we define $\sigma_{\mathsf{p}}(\mathbf{X}) := \sigma_{\mathsf{p}}(C_{\mathbf{X}})$.

Theorem 4.5.6 (Topological HALMOS-VON NEUMANN Theorem)**.** *Assume that G is Abelian. Let $\mathbf{X}, \mathbf{Y} \in \mathsf{EquiDyn}(G)$ be minimal.*

(i) $\mathbf{X} \cong \mathbf{Y}$ *if and only if* $\sigma_{\mathsf{p}}(\mathbf{X}) = \sigma_{\mathsf{p}}(\mathbf{Y})$.

(ii) For any subgroup $\sigma \leq G^*$ there is a minimal $\mathbf{Z} \in \mathsf{EquiDyn}(G)$ with $\sigma_{\mathsf{p}}(\mathbf{Z}) = \sigma$.

(iii) There is a group compactification (K, Ψ) of G, such that $\mathbf{X}$ is conjugate to the rotation on the group compactification as in Definition 1.11.16.[78]

5 Mean Equicontinuity - Topo-Isomorphic MEF

Throughout this section we will assume that: G is a locally compact, left amenable group with a left HAAR measure m. $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ is a left ergodic sequence on G. Fix $(X, G, \alpha) \in \mathsf{CHausDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$.

[78]This is Theorem 4.4.1.

5.1 Metric Case

We follow [Fuhrmann et al., 2018].[79]

In this subsection we additionally assume that X is compact metrizable with a fixed metric d, i.e. $\mathbf{X} \in \mathsf{CMetDyn}(G)$.

Definition 5.1.1. The **$\mathcal{F}$-Besicovitch pseudo-metric** is defined as

$$D_{\mathcal{F}}(x, y) := \limsup_{n\to\infty} \frac{1}{|F_n|} \int_{F_n} d\left(\alpha(g, x), \alpha(g, y)\right) \, \mathrm{d}m(g) \, .$$

Definition 5.1.2. The **Weyl pseudo-metric** is defined as

$$D(x, y) := \sup \left\{ D_{\mathcal{G}}(x, y) \mid \mathcal{G} \text{ is a left ergodic sequence} \right\} \, .$$

Definition 5.1.3 (Continuous Metric)**.** A metric ρ is called **continuous** with respect to the metric d if and only if

$$\forall \varepsilon > 0 : \exists \delta > 0 : d(x, y) < \delta \Longrightarrow \rho(x, y) < \varepsilon \, .$$

Definition 5.1.4 (Besicovitch Mean Equicontinuity)**.** $\mathbf{X}$ is **$\mathcal{F}$-Besicovitch mean equicontinuous** if and only if the $\mathcal{F}$-Besicovitch pseudo-metric $D_{\mathcal{F}}$ is continuous w.r.t. d.

Definition 5.1.5 (Weyl Mean Equicontinuity)**.** $\mathbf{X}$ is **Weyl mean equicontinuous** if and only if the Weyl pseudo-metric D is continuous w.r.t. d.

Theorem 5.1.6. *Suppose that G is σ-compact and $(\mathbf{Y}, \pi)$ is a `mef` of $\mathbf{X}$. The system $\mathbf{X}$ is Weyl-mean equicontinuous if and only if π is a topo-isomorphy.*[80]

Proposition 5.1.7. *Suppose that G is σ-compact and $(\mathbf{Y}, \pi)$ is a `mef` of $\mathbf{X}$. If $\mathbf{X}$ is minimal, then the following are equivalent:*

(i) $\mathbf{X}$ *is Weyl mean equicontinuous.*

(ii) $\mathbf{X}$ *is $\mathcal{F}$-Besicovitch mean equicontinuous.*

(iii) $\mathbf{X}$ *is uniquely ergodic and $\pi : \mathbf{X} \to \mathbf{Y}$ is a topo-isomorphy with respect to the unique invariant probabilities on $\mathbf{X}$ and $\mathbf{Y}$.*[81]

Remark 5.1.7.1. Clearly Weyl mean equicontinuity implies $\mathcal{G}$-Besicovitch mean equicontinuity for any ergodic sequence $\mathcal{G}$. So for minimal systems $\mathcal{F}$-Besicovitch mean equicontinuity for some special ergodic sequence $\mathcal{F}$ implies $\mathcal{G}$-Besicovitch mean equicontinuity for any ergodic sequence $\mathcal{G}$.

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[79] For your inquiries note that in [Fuhrmann et al., 2018] the authors use definitions differing from ours. They call an ergodic sequence as in Definition 1.10.19 a *"Følner sequence"* (Cf. [Fuhrmann et al., 2018, bottom of p.3]).

[80] This is Theorem 1.1 in [Fuhrmann et al., 2018, p.82].

[81] This is Corollary 1.5 in [Fuhrmann et al., 2018].

5.2 Generalizing the Notions to Compact Hausdorff Spaces

In this subsection we assume X to be compact HAUSDORFF, i.e. $\mathbf{X} \in \mathsf{CHausDyn}(G)$. Let m be a **right** HAAR measure and $\mathcal{F}$ a **right** ergodic sequence.

Definition 5.2.1 (Besicovitch Pseudo-Metric (CHaus))**.** We define for any $f \in \mathcal{C}(X, \mathbb{R})$ the f-$\mathcal{F}$-Besicovitch pseudo-metric associated to f and the ergodic sequence $\mathcal{F}$.

$$D_f^{\mathcal{F}}(x, y) := \limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} \left| f(\alpha(g, x)) - f(\alpha(g, y)) \right| \, \mathrm{d}m(g) \,.$$

Remark 5.2.1.1. The Besicovitch Pseudo-Metric is indeed a Pseudo-Metric. Recall that

$$\limsup_{n\to\infty} (a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \,.$$

Using the triangle inequality for the absolute value we can thus conclude that $D_f^{\mathcal{F}}$ satisfies the triangle inequality.

Definition 5.2.2 (Besicovitch mean equicontinuity (CHaus))**.** The system $\mathbf{X}$ is called **$\mathcal{F}$-Besicovitch mean equicontinuous** if and only if $D_f^{\mathcal{F}}$ is continuous for any $f \in \mathcal{C}(X, \mathbb{R})$.

Definition 5.2.3 (Weyl Pseudo-Metric (CHaus))**.** We define the f-Weyl pseudo-metric

$$D_f(x, y) = \sup \left\{ D_f^{\mathcal{G}}(x, y) \,\middle|\, \mathcal{G} \text{ is a right ergodic sequence.} \right\} .$$

Definition 5.2.4 (Weyl mean equicontinuity (CHaus))**.** $\mathbf{X}$ is **Weyl mean equicontinuous** if and only if the pseudo-metric D_f is continuous for any $f \in \mathcal{C}(X, \mathbb{R})$.

Remark 5.2.4.1. By [Fuhrmann et al., 2018, Prop. 3.8] one can see that for compact metric spaces Definition 5.2.4 is equivalent to Definition 5.1.5.

Proposition 5.2.5. *The Besicovitch pseudo-metrics $D_f^{\mathcal{F}}$ are invariant.*

Proof. Let $h \in G$. Note that

$$\int_{F_n} \left| f(\alpha(gh, x)) - f(\alpha(gh, y)) \right| \, \mathrm{d}m(g) = \int_{F_n \cdot h} \left| f(\alpha(g, x)) - f(\alpha(g, y)) \right| \, \mathrm{d}m(g)$$

Choose $C \in \mathbb{R}$ such that $f(x) < C$ for all $x \in X$. Then

$$\begin{aligned} &\left| D_f^{\mathcal{F}}(x, y) - D_f^{\mathcal{F}}(\alpha(h, x), \alpha(h, y) \right| \\ &\leqslant \limsup_{n\to\infty} \frac{1}{m(F_n)} \left| \int_{F_n \triangle F_n \cdot h} \left| f(\alpha(g, x)) - f(\alpha(g, y)) \right| \, \mathrm{d}m(g) \right| \\ &\leqslant \limsup_{n\to\infty} \frac{m(F_n \triangle F_n \cdot h)}{m(F_n)} \cdot C = 0 \,. \end{aligned}$$

□

We define a relation by

$$x \sim y :\Longleftrightarrow \forall f \in \mathcal{C}(X, \mathbb{R}) : D_f^{\mathcal{F}}(x, y) = 0 \,.$$

Lemma 5.2.6. $\sim$ *is an* `icer`*.*

Proof. Note that $\sim = \bigcap_f D_f^{-1}(\{0\})$. So $\sim$ is closed. By Proposition 5.2.5 $\sim$ is invariant. Now use the triangle inequality to conclude that $\sim$ is an equivalence relation. □

Definition 5.2.7 (Regionally Proximal Relation)**.** Define a relation Q by

$$xQy :\Leftrightarrow \forall \varepsilon \in \mathcal{U}(\Delta) : \exists g_\varepsilon \in G : \exists x_\varepsilon \in \varepsilon[x] : \exists y_\varepsilon \in \varepsilon[y] : (\alpha(g_\varepsilon, x_\varepsilon), \alpha(g_\varepsilon, y_\varepsilon)) \in \varepsilon\,.$$

We call Q the **regionally proximal relation**.

Q is not always an equivalence relation but:

Theorem 5.2.8. *Let Q^* be the smallest (by inclusion)* `icer` *containing Q. The system $\mathbf{X}/Q^*$ is a* `mef` *of $\mathbf{X}$.*[82]

Proposition 5.2.9. *Let $\pi : X \to X/{\sim}$ be the quotient map. If $\mathbf{X}$ is $\mathcal{F}$-Besicovitch mean equicontinuous then $(\mathbf{X}/{\sim}, \pi)$ is a* `mef` *of $\mathbf{X}$.*

Proof. If we show that $\sim = Q$ we can conclude that Q is an `icer`. This yields that $\mathbf{X}/Q = \mathbf{X}/{\sim}$ is a `mef`. So let xQy. $\mathbf{X}$ is $\mathcal{F}$-Besicovitch mean equicontinuous, so $D_f^{\mathcal{F}}$ is invariant and continuous for any $f \in \mathcal{C}(X, \mathbb{R})$. Note that we can view $\mathcal{U}(\Delta)$ as a directed set via inclusion and thus obtain a net. Calculate

$$\begin{aligned} D_f^{\mathcal{F}}(x,y) &= \lim_{\varepsilon \in \mathcal{U}(\Delta)} D_f^{\mathcal{F}}(x_\varepsilon, y_\varepsilon) \\ &= \lim_{\varepsilon \in \mathcal{U}(\Delta)} D_f^{\mathcal{F}}\left(\alpha(g_\varepsilon, x_\varepsilon), \alpha(g_\varepsilon, y_\varepsilon)\right)\,. \end{aligned}$$

As $D_f^{\mathcal{F}}$ is continuous the latter is zero.[83] Thus $D_f^{\mathcal{F}}(x,y) = 0$. As f was arbitrary we learn $x \sim y$.

Now suppose that $(x,y) \notin Q$. Then there is an $\varepsilon \in \mathcal{U}(\Delta)$ such that

$$\forall g \in G : \forall x' \in \varepsilon[x] : \forall y' \in \varepsilon[y] : (\alpha(g,x'), \alpha(g,y')) \notin \varepsilon\,.$$

Choose $\delta \in \mathcal{U}(\Delta)$ such that $\mathrm{cl}(\delta) \subseteq \varepsilon$. By the Urysohn's Lemma 1.7.2 there is a continuous function $f : X \to [0,1]$ separating ε^c and $\mathrm{cl}(\delta)$, such that $f|_{\varepsilon^c} = 1$ and $f|_{\mathrm{cl}(\delta)} = 0$. Clearly $|f(x) - f(y)| < 1$ implies that $(x,y) \in \varepsilon$. As $\alpha(g,x), \alpha(g,y) \notin \varepsilon$ we conclude that $|f(\alpha(g,x)) - f(\alpha(y,x))| = 1$. We calculate

$$\begin{aligned} D_f^{\mathcal{F}}(x,y) &= \limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} \left| f(\alpha(g,x)) - f(\alpha(g,y)) \right| \,\mathrm{d}m(g) \\ &= \limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} 1 \,\mathrm{d}m(g) \\ &= \limsup_{n\to\infty} \frac{m(F_n)}{m(F_n)} = 1 > 0\,. \end{aligned}$$

This yields that $D_f^{\mathcal{F}}(x,y) = 1$ and thus $x \not\sim y$. □

Corollary 5.2.9.1. *Let $\rho : \mathbf{X} \to \mathsf{MEF}(\mathbf{X})$ be any universal factor map. For $x, y \in X$ we have*

$$\forall f \in \mathcal{C}(X, \mathbb{R}) : D_f^{\mathcal{F}}(x,y) = 0 \Longleftrightarrow \rho(x) = \rho(y)\,.$$

[82] This is Theorem 3 from Chapter 9 in [Auslander, 1988, p.127].

[83] Suppose that $D_f^{\mathcal{F}}\left(\alpha(g_\varepsilon, x_\varepsilon), \alpha(g_\varepsilon, y_\varepsilon)\right)$ does not converge to zero. Then there is a subnet bounded away from zero. Pick a subnet of this subnet such that $\alpha(g_\varepsilon, x_\varepsilon)$ converges with limit $z \in X$. For the same subnet $\alpha(g_\varepsilon, y_\varepsilon)$ also converges to z. This is a contradiction.

5.3 Back to the Metric Case

We can translate Proposition 5.2.9 into

Proposition 5.3.1. *Let X be a compact metric space and $\rho : \mathbf{X} \to \mathsf{MEF}(\mathbf{X})$ be any universal factor map. For $x, y \in X$ we have*

$$D_{\mathcal{F}}(x,y) = 0 \iff \rho(x) = \rho(y)\,.$$

Proof. We already know that $\rho(x) = \rho(y)$ is equivalent to the fact that $D_f^{\mathcal{F}}(x,y) = 0$ for any $f \in \mathcal{C}(X,\mathbb{R})$.

Let L^* denote the family of Lipschitz continuous functions and L the subfamily of those with Lipschitz constant 1. Now $|f(x) - f(y)| \leqslant d(x,y)$ for any $f \in L$. By the triangle inequality we have that $f_x \in L$ where $f_x(z) := d(x,z)$. Hence, $d(x,y) = \sup_{f \in L} |f(x) - f(y)|$.

Now suppose that $D_{\mathcal{F}}(x,y) = 0$. We must show that $D_f^{\mathcal{F}}(x,y) = 0$ for any f. Further observe that by the triangle inequality and the continuity of integrals w.r.t. uniform convergence that $D_f^{\mathcal{F}}(x,y)$ is continuous in f w.r.t. the uniform topology.

As L^* is uniformly dense in $\mathcal{C}(X,\mathbb{R})$ it suffices to consider $f \in L^*$. Let $f \in \mathcal{C}(X,\mathbb{R})$ be Lipschitz with constant K. Now see that

$$\begin{aligned} D_f^{\mathcal{F}}(x,y) &\leqslant \limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} K \cdot d(\alpha(g,x), \alpha(g,y))\, \mathrm{d}m(g) \\ &= K \cdot D_{\mathcal{F}}(x,y) = 0\,. \end{aligned}$$

Conversely, suppose that $D_{\mathcal{F}}(x,y) > 0$. Let $0 < 2\varepsilon < D_{\mathcal{F}}(x,y)$ and choose a finite family of functions $\{f_1, \ldots, f_n\}$ such that whenever $d(x,y) > \varepsilon$ there is $i \in \{1, \ldots, n\}$ such that $|f_i(x) - f_i(y)| > 1$. One way to construct such a family is by choosing a finite $\frac{\varepsilon}{2}$-dense subset $\{x_1, \ldots, x_n\}$. Then use Urysohn's Lemma 1.7.2 in order to obtain functions f_i such that $f_i|_{\mathrm{cl}(B_{\frac{\varepsilon}{2}}(x_i))} = 1$ and $f_i|_{B_\varepsilon(x_i)^c} = 0$. By the $\frac{\varepsilon}{2}$-denseness of the x_i any $x \in X$ lies in some of the $B_{\frac{\varepsilon}{2}}(x_i)$. Further if $d(x,y) > \varepsilon$ then $y \in B_\varepsilon(x_i)^c$. Now $D_{\mathcal{F}}(x,y) > 2\varepsilon$ implies that $d(\alpha(g,x), \alpha(g,y)) > 0$ with positive $\mathcal{F}$-density. Thus $\exists i \in \{1, \ldots, n\} : |f_i(x) - f_i(y)| > 1$ with positive upper $\mathcal{F}$-density. As the family of functions is finite and the upper $\mathcal{F}$-density is finitely subadditive we conclude that there is $i \in \{1, \ldots, n\}$ such that $|f_i(x) - f_i(y)| > 1$ with positive upper $\mathcal{F}$-density. This means that $D_{f_i}^{\mathcal{F}}(x,y) > 0$. □

5.4 Mean Equicontinuity implies Weak-Topo-Isomorphy

We generalize one implication of the equivalence stated in Theorem 5.1.6 to a particular class of compact Hausdorff spaces. In the process we develop a toolbox for working in the more general setting of compact Hausdorff spaces. One reason to choose that setting is that the category of compact Hausdorff spaces is in many ways much better behaved than the category of compact metric spaces. For example arbitrary products of compact Hausdorff spaces remain compact Hausdorff.

In this subsection we have the following **standing assumptions**: We denote the diagonal of the Cartesian square of a topological space Z by Δ_Z, i.e.

$$\Delta_Z := \{(z,z) \mid z \in Z\} \subseteq Z \times Z\,.$$

Let $\mathfrak{A}$ and $\mathfrak{F}$ be the BOREL-σ-algebras on X and Y respectively. The phase space X is compact HAUSDORFF. The diagonal $\Delta_Y \subseteq Y \times Y$ is measurable with respect to the product-σ-algebra of the BOREL-σ-algebra on Y.[84] This means that $\Delta_Y \in \mathfrak{F} \otimes \mathfrak{F}$.[85] Further the diagonals Δ_Y and Δ_X are contained in the BAIRE-σ-algebra on $Y \times Y$ and $X \times X$ respectively.[86] The group G shall be locally compact and right amenable. m is a right HAAR measure on G and $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ is a right ergodic sequence.

It is the task of future investigations to get rid of some of those assumptions or to find counterexamples. Clearly those assumptions are very strong and the category of spaces satisfying those assumptions behaves as badly as the one of compact metric spaces.

Theorem 5.4.1. *Let $(\mathbf{Y}, \pi)$ be a* `mef` *of $\mathbf{X}$. If $\mathbf{X}$ is $\mathcal{F}$-Besicovitch mean equicontinuous, then $\pi : \mathbf{X} \to \mathbf{Y}$ is a weak-topo-isomorphism.*

Proof. Let $(Y, G, \beta) = \mathbf{Y}$. Consider the space

$$\Xi := X \times_\pi X = \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\} \, .$$

Note that $\Xi = (\pi \times \pi)^{-1}(\Delta_Y)$. Y is HAUSDORFF so Δ_Y is closed. Thus, Ξ is closed and thus compact. Clearly Ξ is $\alpha \times \alpha$-invariant.

We show that the graph of π is measurable, i.e.

$$\mathrm{Graph}(\pi) := \{(x, \pi(x)) \mid x \in X\} = \{(x, y) \in X \times Y \mid \pi(x) = y\} \in \mathfrak{A} \otimes \mathfrak{F} \, ,$$

Note that

$$\pi \otimes \mathrm{Id}_Y : X \times Y \longrightarrow Y \times Y, \ (x, y) \longmapsto (\pi(x), y)$$

satisfies $(\pi \otimes \mathrm{Id}_Y)^{-1}(\Delta_Y) = G(\pi)$. π is $\mathfrak{A}$-$\mathfrak{F}$-measurable and Id_Y is $\mathfrak{F}$-measurable. So $\pi \otimes \mathrm{Id}_Y$ is $\mathfrak{A} \otimes \mathfrak{F}$-$\mathfrak{F} \otimes \mathfrak{F}$-measurable. The standing assumption that $\Delta_Y \in \mathfrak{F} \otimes \mathfrak{F}$ implies that $\mathrm{Graph}(\pi) \in \mathfrak{A} \otimes \mathfrak{F}$.

Fix a regular invariant probability measure μ on $\mathfrak{B}X$. By Theorem 1.8.19 and 1.8.20 there is a regular conditional probability $R : \mathfrak{F} \times Y \to [0, 1]$ w.r.t. π such that

$$\forall y \in Y : R\left(\pi^{-1}\left(\{y\}\right), y\right) = 1 \, . \tag{17}$$

Let $\mathfrak{M}$ be the BOREL-σ-algebra on $X \times X$ and define a probability

$$\mu \times_\pi \mu : \mathfrak{M} \longrightarrow [0, 1], A \longmapsto \int \left(R(\cdot, y) \otimes R(\cdot, y)\right)(A) \, \mathrm{d}\pi_* \mu \, .$$

Observe that $\pi^{-1}\{y\} \times \pi^{-1}\{y\} \subseteq \Xi$ for all $y \in Y$. Equation (17) implies $\mu \times_\pi \mu(\Xi) = 1$. As $\mu \times_\pi \mu$ may not be regular we do a trick: By the Riesz Representation Theorem 1.8.17, there is a unique regular probability $\rho : \mathfrak{M} \to [0, 1]$ such that

$$\int f \, \mathrm{d}\rho = \int f \, \mathrm{d}\mu \times_\pi \mu \tag{18}$$

[84] This is satisfied for example if Y is compact metric.

[85] Note that the measurability of the diagonal enforces that $\mathsf{card}(Y) \leqslant 2^{\aleph_0}$ as pointed out by Eric Wofsey in his post [Wofsey, 2018]. This is due to the fact that for any fixed $A \in \sigma(Y) \otimes \sigma(Y)$ we have that $\mathsf{card}\{A_y \mid y \in Y\} \leqslant 2^{\aleph_0}$, where $A_y = \{y' \in Y \mid (y', y) \in A\}$. This fact is easily proved by observing that $\left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$. This implies that the set S of all A such that $\mathsf{card}\{A_y \mid y \in Y\} \leqslant 2^{\aleph_0}$ is a σ-algebra. As clearly $\sigma(Y) * \sigma(Y) \subseteq S$ we see that $\sigma(Y) \otimes \sigma(Y) \subseteq S$.

[86] This is satisfied for example if X and Y are perfectly normal (as then any BOREL measurable set is BAIRE measurable).

for any $f \in \mathcal{C}(X \times X, \mathbb{R})$. Let $\mathfrak{h}$ be the system of all measurable functions

$$f : X \times X \longrightarrow [0,1]$$

satisfying (18). The Lebesgue Convergence Theorem 1.6.21 implies that $\mathfrak{h}$ is closed under pointwise convergence. By Corollary 1.8.46.2 $\mathfrak{h}$ contains all BAIRE measurable functions.[87] We assume that Δ_Y is BAIRE measurable. So $\Xi = (\pi \times \pi)^{-1}(\Delta_Y)$ is BAIRE measurable. Thus $\rho(\Xi) = \mu \times_\pi \mu(\Xi) = 1$. So we view ρ as a probability measure on Ξ.

Suppose $\rho(\Delta_X) < 1$. By regularity there is a compact subset $K \subseteq \Delta_X^c$ such that $\rho(K) > 0$. With Corollary 1.12.13.2 we can find a compact BAIRE measurable set $B \supseteq K$ disjoint from the diagonal and an ergodic probability $\tilde{\rho}$ such that $\tilde{\rho}(B) > 0$. As $B \cap \Delta_X = \emptyset$ we have $p_1(B) \cap p_2(B) = \emptyset$. By URYSOHN's Lemma 1.7.2 find a continuous function $f : X \to [0,1]$ such that $f|_{p_1(B)} = 1$ and $f|_{p_2(B)} = 0$. Then $H(x,y) := |f(x) - f(y)| > 0$ for any $(x,y) \in B$. So $\int H \, \mathrm{d}\tilde{\rho} > 0$. The Mean Ergodic Theorem 1.12.6 implies

$$\frac{1}{m(F_n)} \int_{F_n} H \circ (\alpha \times \alpha)(g, \cdot) \, \mathrm{d}m(g) \xrightarrow[n\to\infty]{L^1} \int H \, \mathrm{d}\tilde{\rho} \,.$$

By Lemma 1.6.20 there is a $n_k \xrightarrow{k\to\infty} \infty$, defining a subsequence, such that

$$\frac{1}{m(F_{n_k})} \int_{F_{n_k}} H \circ (\alpha \times \alpha(g, \cdot) \, \mathrm{d}m(g) \xrightarrow[k\to\infty]{\tilde{\rho}\text{-almost surely}} \int H \, \mathrm{d}\tilde{\rho} \,.$$

As $\tilde{\rho}(B) > 0$, there is $(a,b) \in B$ such that

$$\begin{aligned} D_f^{\mathcal{F}}(a,b) &\geqslant \limsup_{k\to\infty} \frac{1}{m(F_{n_k})} \int_{F_{n_k}} \big|f(\alpha(g,a)) - f(\alpha(g,b))\big| \, \mathrm{d}m(g) \\ &= \int H \, \mathrm{d}\tilde{\rho} > 0 \,. \end{aligned}$$

However, $\pi(a) = \pi(b)$ as $(a,b) \in \Xi$. Thus $D_f^{\mathcal{F}}(a,b) = 0$ by Corollary 5.2.9.1. Thus, $\rho(\Delta_X) = 1$. As the diagonal is assumed to be BAIRE measurable we have

$$(\mu \times_\pi \mu)(\Delta_X) = \rho(\Delta_X) = 1 \,.$$

So $(R(\cdot,y) \otimes R(\cdot,y))(\Delta_X) = 1$ $\pi_*\mu$-almost surely. Suppose there are disjoint sets $A, B \subseteq \pi^{-1}(\{y\})$ such that $R(A,y) > 0$ and $R(B,y) > 0$. Then

$$(R(\cdot,y) \otimes R(\cdot,y))(\Delta^c) \geqslant (R(\cdot,y) \otimes R(\cdot,y))(A \times B) = R(A,y) \cdot R(B,y) > 0$$

Thus for $\pi_*\mu$-almost all y there cannot be two disjoint sets A, B such that $R(A,y) > 0$ and $R(B,y) > 0$. Using Proposition 1.8.7 we conclude that, there is a measurable set $Y_0 \subseteq Y$ with $\pi_*\mu(Y_0) = 1$ such that there is $x_y \in X$ with $R(\cdot,y) = \delta_{x_y}$ for any $y \in Y_0$.

[87]Note that this yields a proof to Theorem 1.12.14. This theorem extends ULAM's Theorem 1.8.9 as for metric spaces the BOREL- and the BAIRE-σ algebra coincide.

Now define $\mathcal{G} : Y_0 \to X,\ y \mapsto x_y$. Recall that $\mathfrak{F}_\mu$ denotes the completion of $\mathfrak{F}$ w.r.t. μ. By Definition 1.8.18 *(b)* the function $R(A, \cdot)$ is $\mathfrak{F}_\mu$-measurable. So

$$\mathcal{G}^{-1}(A) = \{y \in Y_0 \mid \mu_y(A) = 1\} = Y_0 \cap R(A, \cdot)^{-1}(\{1\}) \in \mathfrak{F}_\mu$$

for any $A \in \mathfrak{A}$. Therefore, $\mathcal{G}$ is an $\mathfrak{F}_\mu$-measurable map. Clearly,

$$\mathcal{G}(y) \in \operatorname{supp}(R(\cdot, y)) \subseteq \pi^{-1}(\{y\})$$

for $y \in Y_0$. This shows

$$\pi \circ \mathcal{G} = \mathrm{Id}_{Y_0}\,. \tag{19}$$

Define $X_0 = \mathcal{G}(Y_0)$. Let $\mathfrak{A}_y$ be the completion of $\mathfrak{A}$ w.r.t. $R(\cdot, y)$ and let $R^*(\cdot, y)$ denote the unique extension of $R(\cdot, y)$ to $\mathfrak{A}_y$. By definition $x_y \in X_0$ for $y \in Y_0$. Thus $X_0 \in \mathfrak{A}_y$ and $R^*(X_0, y) = 1$ for any $y \in Y_0$. In particular $R^*(X_0, \cdot)$ is $\mathfrak{A}_y$-measurable on Y_0. We conclude $X_0 \in \mathfrak{A} \dagger R$. Furthermore, $R^*(A \cap X_0, y) = R(A, y)$ for any $y \in Y_0$. So $R^*(A \cap X_0, \cdot)$ is $\mathfrak{A}_y$-measurable and $X_0 \cap A \in \mathfrak{A} \dagger R$ for $A \in \mathfrak{A}$.

Lemma 1.8.24 implies that $\mathfrak{A} \dagger R$ is a Dynkin system. By the Dynkin-π-λ-Theorem 1.6.5 the σ-algebra $\mathfrak{F}$ generated by $\{X_0\} \cup \mathfrak{A}$ is contained in $\mathfrak{A} \dagger R$. With Definition 1.8.26 we can extend μ to a probability μ^* on $\mathfrak{F}$. Calculate

$$\mu^*(X_0) = \int_Y R^*(X_0, y)\, \mathrm{d}\pi_*\mu(y) \geqslant \int_{Y_0} R^*(X_0, y)\, \mathrm{d}\pi_*\mu(y) = \int_{Y_0} 1\, \mathrm{d}\pi_*\mu = 1\,.$$

Now let $x \in X_0$. By definition there is $y \in Y_0$ such that $x = \mathcal{G}(y)$. So we calculate

$$\mathcal{G}(\pi(x)) = \mathcal{G}(\pi(\mathcal{G}(y))) = \mathcal{G}(y) = x\,.$$

Conclude $\mathcal{G} \circ \pi|_{X_0} = \mathrm{Id}_{X_0}$. Together with (19) we see $\pi|_{X_0} = \mathcal{G}^{-1}$. So $\mathcal{G}$ is a weak isomorphism $\mathcal{G} : (\mathfrak{B}Y, \pi_*\mu) \to (\mathfrak{B}X, \mu)$. As μ was arbitrary this shows that π is a weak-topo-isomorphy. □

Remark 5.4.1.1. Note that π induces an isomorphism of the measure algebras $\mathsf{ma}(\mathfrak{A}, \mu)$ and $\mathsf{ma}(\mathfrak{F}, \pi_*\mu)$ by Corollary 1.8.34.1.

6 Frequent Stability - Topo-Isomorphic and almost 1-1 MEF

Throughout this section we assume the following: G is a locally compact, left amenable group and m is a left Haar measure on G. Further $\mathcal{F}$ is a left ergodic sequence on G. The phase space $X \neq \emptyset$ is compact metrizable and d_X is a metric on X. Fix $(X, G, \alpha) \in \mathsf{CMetDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$.

The following presentation is taken from the paper [García-Ramos et al., 2021, Section 3] but generalized to the setting of amenable acting groups.

Definition 6.0.1. A point $x \in X$ is **($\mathcal{F}$-)frequently stable**, if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$A_{\delta,\varepsilon} := \{g \in G \mid \mathsf{diam}\,(\alpha(g, B_\delta(x))) > \varepsilon\}$$

has an upper density less than 1, i.e. $\bar{D}_{\mathcal{F}}(A_{\delta,\varepsilon}) < 1$.

Theorem 6.0.2. *Let* $\mathbf{X}$ *be minimal and mean equicontinuous and* $(\mathbf{Y}, \pi)$ *a* `mef` *of* $\mathbf{X}$*. Let* G *be a* σ*-compact and locally compact* HAUSDORFF *group. For a fixed ergodic sequence* $\mathcal{F}$ *the following are equivalent:*

(i) $\pi : \mathbf{X} \to \mathbf{Y}$ *is almost 1-1.*

(ii) Every $x \in X$ *is* $\mathcal{F}$*-frequently stable.*

(iii) There is at least one $\mathcal{F}$*-frequently stable point* $x \in X$*.*[88]

Remark 6.0.2.1. Note that the invertibility property *(i)* is independent of the ergodic sequence $\mathcal{F}$. Thus, frequent stability with respect to one ergodic sequence is equivalent to frequent stability with respect to any other ergodic sequence.

The metric d_Y on Y can be w.l.o.g. assumed to be β-invariant. Also note that by Proposition 5.1.7 $\mathbf{X}$ is uniquely ergodic. Let the unique invariant probability be denoted by μ. The unique invariant probability on $\mathbf{Y}$ will be denoted by ν. The proof of the implication from *(iii)* to *(i)* is quite technical and opaque so let us discuss the strategy first to gain some insight.

Strategy. By contraposition we show that if π is not almost 1-1 then no point is $\mathcal{F}$-frequently stable. So let π not be almost 1-1. Then every point $y \in Y$ has multiple preimages and the size of the fibers is bounded below by some $\varepsilon > 0$.

As π is a topo-isomorphy there is an almost surely defined measurable inverse $\rho : Y_0 \to X_0$. From $\rho_* \nu = \mu$ we learn $\mu(X_0) = 1$. For any $\eta > 0$ and $y \in Y$ we have that $\mathsf{diam}\left(\rho(B_\eta(y))\right) > \varepsilon$. We can even that there are $B_1^y, B_2^y \subseteq X_0$ with $d_X(B_1^y, B_2^y) \geqslant \frac{\varepsilon}{4}$ such that a point in $B_\eta(y)$ is mapped to B_i^y with probability $\kappa > 0$.

Now fix an $x \in X$ and any $\delta > 0$. There is $y \in Y$ and $\eta > 0$ such that $B_\eta(y) \subseteq \pi\left(B_\delta(x)\right)$. By the Jankov-von Neumann Selection Theorem 1.8.13 we find $\varphi : B_\eta(y) \to B_\delta(x)$ which is a local left-inverse of π.

Now we show that the diameter of $\alpha(g, \varphi(B_\eta(y)))$ is very close to the diameter of $\rho(\beta(g, B_\eta(y)))$ for $g \in H$, where $H \subseteq G$ has positive density. In particular we will show $\mathsf{diam}\left(\rho(\beta(g, B_\eta(y)))\right) > \frac{\varepsilon}{8}$ for $g \in H$. As we made sure that $\varphi(B_\eta(y)) \subseteq B_\delta(x)$ this disproves the $\mathcal{F}$-frequent stability of x. □

Proof. As X is non-empty *(ii)* implies *(iii)*. We show that *(i)* implies *(ii)* and that *(iii)* implies *(i)*.

***(i)* implies *(ii)*:** Let $\varepsilon > 0$ and $x \in X$ be arbitrary. We start by finding an easier to handle condition which implies that $\mathsf{diam}(\alpha(g, B_\delta(x))) < \varepsilon$.

As π is almost 1-1, there is at least one $y_0 \in Y$ such that $\mathsf{card}\left(\pi^{-1}\left(\{y_0\}\right)\right) = 1$. By Lemma 3.1.4 we know that there is $\eta > 0$ such that $\mathsf{diam}\left(\pi^{-1}\left(B_\eta(y)\right)\right) < \varepsilon$ for any $y \in B_\eta(y_0)$. The factor map π is continuous. Thus there is $\delta > 0$ such that $\pi\left(B_\delta(x)\right) \subseteq B_\eta(\pi(x))$. Further, β is an isometry and thus

[88] This is a straightforward generalization of the statement and proof of Theorem 3.4 provided in [García-Ramos et al., 2021].

$B_\eta\left(\beta(g,\pi(x))\right) = \beta\left(g, B_\eta(\pi(x))\right)$ for any $g \in G$. Now π is equivariant and we can thus deduce

$$\begin{aligned}\pi\left(\alpha\left(g, B_\delta(x)\right)\right) &= \beta\left(g, \pi\left(B_\delta(x)\right)\right)\\ &\subseteq \beta\left(g, B_\eta(\pi(x))\right)\\ &= B_\eta\left(\beta(g,\pi(x))\right)\end{aligned}$$

for all $g \in G$. Taking the preimage we further obtain

$$\alpha\left(g, B_\delta(x)\right) \subseteq \pi^{-1}\left(B_\eta\left(\beta(g,\pi(x))\right)\right) . \tag{20}$$

Now assume that $\beta(g,\pi(x)) \in B_\eta(y_0)$. Then $\mathsf{diam}\left(\pi^{-1}\left(B_\eta\left(\beta(g,\pi(x))\right)\right)\right) < \varepsilon$. Equation (20) yields further $\mathsf{diam}\left(\alpha\left(g, B_\delta(x)\right)\right) < \varepsilon$. In total

$$\{g \in G \mid \beta(g,\pi(x)) \in B_\eta(y_0)\} \subseteq \{g \in G \mid \mathsf{diam}\left(\alpha\left(g, B_\delta(x)\right)\right) < \varepsilon\} .$$

It is thus sufficient to show that the former has positive lower density.

Y inherits the minimality from **X**. Observe that $\mathrm{supp}(\nu) = Y$. In particular, $\nu\left(B_\rho(y_0)\right) > 0$ for any $\rho > 0$. Fix $\eta > \rho > 0$. There is a continuous function $f \in \mathcal{C}(Y,[0,1])$ such that $f|_{\mathrm{cl}(B_\rho)(y_0)} = 1$ and $f|_{B_\eta(y_0)^c} = 0$. The Uniform Ergodic Theorem 1.13.5 applied to the ergodic sequence $\mathcal{F}$ shows that for all $y \in Y$ we have

$$\begin{aligned}\underline{D}^{\mathcal{F}}\left(\{g \in G \mid \beta(g,y) \in B_\eta(y_0)\}\right) &\geqslant \underline{D}^{\mathcal{F}}\left(\{g \in G \mid f(\beta(g,y)) > 0\}\right)\\ &\geqslant \int f \,\mathrm{d}\nu\\ &> \nu\left(B_\rho(y_0)\right)\\ &> 0 .\end{aligned}$$

***(iii)* implies *(i)*:** We proceed by contraposition, and show that that if π is not almost 1-1 then no $x \in X$ is frequently stable. For that we find an $\varepsilon > 0$ and a set $H \subseteq G$ with $\bar{D}_{\mathcal{F}}(H) = 1$ such that $\mathsf{diam}(\alpha\left(g, B_\delta(x)\right)) > \frac{\varepsilon}{4}$ for $g \in H$.

By rescaling the metric, we can w.l.o.g. assume $\mathsf{diam}(X) = 1$. Let $\mathfrak{A}$ be the Borel-σ-algebra on X. **X** is assumed to be mean equicontinuous. By Proposition 5.1.7 π is a topo-isomorphy with respect to the unique invariant probabilities. There are $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $\mu(X_0) = 1 = \nu(Y_0)$, such that: $\pi|_{X_0} : X_0 \to Y_0$ is bijective with measurable inverse ρ and $\pi_*\mu = \nu$ and $\rho_*\nu = \mu$. Thus,

$$\mu(A) = \nu\left(\rho^{-1}(A)\right)$$

for any $A \in \mathfrak{A}$.

Let $\delta > 0$ be arbitrary. By Lemma 3.5.5 there is $\varepsilon > 0$ such that $\pi_*\mathsf{diam}(y) > \varepsilon$ for all $y \in Y$. The continuity of π yields that

$$\mathrm{Graph}(\pi) = \{(x,y) \in X \times Y \mid \pi(x) = y\} \subseteq X \times Y$$

is closed and hence compact. For any $\eta > 0$ define the map

$$\Psi_\eta : \mathrm{Graph}(\pi) \longrightarrow \mathbb{R},\ (x,y) \longmapsto \mu\left(B_{\frac{\varepsilon}{8}}(x) \cap \pi^{-1}\left(B_\eta(y)\right)\right) .$$

Assume that the infimum of Ψ_η is zero. Then there is a minimizing sequence $(x_n, y_n)_{n\in\mathbb{N}} \in \mathrm{Graph}(\pi)^{\mathbb{N}}$ i.e.

$$\lim_{n\to\infty} \Psi_\eta(x_n, y_n) = 0 .$$

If necessary we can choose convergent sub-sequences, thus we can w.l.o.g. assume that there are $x \in X$ and $y \in Y$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y .$$

The open set $U := B_{\frac{\varepsilon}{16}}(x) \cap \pi^{-1}(B_{\frac{\eta}{2}}(y))$ has positive measure. Pick $N \in \mathbb{N}$ such that for $n > N$ we have $d_X(x_n, x) < \frac{\varepsilon}{16}$ and $d_Y(y_n, y) < \frac{\eta}{2}$. Then $U \subseteq B_{\frac{\varepsilon}{8}}(x_n) \cap \pi^{-1}\left(B_\eta(y_n)\right)$ for $n > N$. We conclude

$$0 < \mu(U) \leqslant \mu\left(B_{\frac{\varepsilon}{8}}(x_n) \cap \pi^{-1}\left(B_\eta(y_n)\right)\right) = \Psi_\eta(x_n, y_n) .$$

This is a contradiction. We have proved the existence of a $\kappa > 0$ such that $\Psi_\eta(x,y) > \kappa$ for all $(x,y) \in \mathrm{Graph}(\pi)$.

By Lemma 3.1.3 there is an $\eta > 0$ such that for any $x \in X$ there is a $y_x \in Y$ such that $B_\eta(y_x) \subseteq \pi(B_\delta(x))$. Let $x \in X$ be arbitrary and set $y := y_x$. Denote $L := B_\delta(x) \cap \pi^{-1}(B_\eta(y))$. Then $\pi|_L : L \to B_\eta(y)$ is surjective. Now observe that

$$\Gamma := \mathrm{Graph}(\pi|_L) = \mathrm{Graph}(\pi) \cap (L \times Y)$$

is Borel measurable in $X \times Y$. Clearly $s : X \times Y \to Y \times X$, $(x,y) \mapsto (y,x)$ is a homeomorphism. Thus s is bi-measurable. Hence,

$$\Gamma^{-1} = \{(y,x) \mid (x,y) \in \Gamma\} \subset Y \times X$$

is also Borel measurable. Remark 1.8.11.1 states that all measurable sets in Borel spaces are analytic. So the set Γ^{-1} is analytic. By the Jankov-von Neumann Selection Theorem 1.8.13 there exists an analytically measurable function

$$\varphi : \mathrm{proj}_Y(\Gamma^{-1}) \to X$$

such that $\mathrm{Graph}(\varphi) \subseteq \Gamma^{-1}$. This means that $\varphi : B_\eta(y) \to B_\delta(x)$ is a right inverse of π, i.e.

$$\forall y' \in B_\eta(y) : \pi \circ \varphi(y') = y' .$$

As φ is a choice function for the fibers just as ρ we have that $\pi(\varphi(y')) = \pi(\rho(y'))$ for all $y' \in B_\eta(y) \cap Y_0$. Thus, Proposition 5.3.1 implies that

$$\lim_{n\to\infty} \frac{1}{F_n} \int_{F_n} d_X\left(\alpha(g, \varphi(y')), \alpha(g, \rho(y'))\right)\, \mathrm{d}m(g) = D_{\mathcal{F}}(\varphi(y'), \rho(y')) = 0 .$$ [89]

[89] Note that the integral is well-defined as φ is analytically measurable and hence universally measurable by Theorem 1.8.15. Thus φ is measurable with respect to the completion of $\mathfrak{A}$ w.r.t. μ which means the integral is well-defined.

It holds $\mathsf{diam}(X) = 1$ so $d_X(x, x') \leqslant 1$. Hence, $\mathbb{1}_Y$ is an integrable majorant of

$$\left(y' \longmapsto \frac{1}{F_n} \int_{F_n} d_X \left(\alpha(g, \varphi(y')), \alpha(g, \rho(y'))\right) \, \mathrm{d}m(g)\right)_{n \in \mathbb{N}} .$$

This yields, that we can apply the dominated convergence theorem and obtain

$$\lim_{n \to \infty} \int_{B_\eta(y)} \frac{1}{F_n} \int_{F_n} d_X \left(\alpha(g, \varphi(y')), \alpha(g, \rho(y'))\right) \, \mathrm{d}m(g) \, \mathrm{d}\nu(y') = 0 . \tag{21}$$

For $g \in G$ we define

$$E_g := \left\{ y' \in B_\eta(y) \cap Y_0 \;\middle|\; d_X(\alpha(g, \varphi(y')), \alpha(g, \rho(y'))) \leqslant \frac{\varepsilon}{8} \right\} .$$

An easy estimation yields that

$$\begin{aligned} 0 &= \lim_{n \to \infty} \int_{B_\eta(y)} \frac{1}{F_n} \int_{F_n} d_X \left(\alpha(g, \varphi(y')), \alpha(g, \rho(y'))\right) \, \mathrm{d}m(g) \, \mathrm{d}\nu(y') \\ &\geqslant \frac{\kappa}{2} \cdot \frac{\varepsilon}{8} \cdot \bar{D}_{\mathcal{F}} \left(\left\{ g \in G \;\middle|\; \nu\left(B_\eta(y) \setminus E_g\right) \geqslant \frac{\kappa}{2} \right\} \right) \geqslant 0 . \end{aligned}$$

So we can calculate with (21) that

$$\bar{D}_{\mathcal{F}} \left(\left\{ g \in G \;\middle|\; \nu\left(B_\eta(y) \setminus E_g\right) \geqslant \frac{\kappa}{2} \right\} \right) = 0 .$$

By Lemma 1.10.25 the upper density map $\bar{D}_{\mathcal{F}}$ is subadditive. We conclude that $\bar{D}_{\mathcal{F}}(H) = 1$, where

$$\begin{aligned} H :&= \left\{ g \in G \;\middle|\; \nu(E_g) > \nu(B_\eta(y)) - \frac{\kappa}{2} \right\} \\ &= \left\{ g \in G \;\middle|\; \nu\left(B_\eta(y) \setminus E_g\right) \geqslant \frac{\kappa}{2} \right\}^c \end{aligned} \tag{22}$$

Now we show that for $g \in H$ the condition for frequent stability can't be satisfied. Recall that $\pi_* \mathsf{diam}(y) > \varepsilon$ for all $y \in Y$. This means that for any $y \in Y$ there is $x_1^y, x_2^y \in \pi^{-1}(\{y\})$ such that $d_X(x_1^y, x_2^y) > \frac{\varepsilon}{2}$. Define

$$B_j^y := B_{\frac{\varepsilon}{8}}(x_j^y) \cap X_0 \cap \pi^{-1}\left(B_\eta(y) \cap Y_0\right) \subseteq X_0 .$$

By the above we have $\mu(B_j^y) > \kappa$. Defining $A_j^y := \pi(B_j^y)$ we further have $\nu(A_j^y) > \kappa$. Further clearly $d_X(B_1^y, B_2^y) \geqslant \frac{\varepsilon}{4}$. Finally note that $A_j^y \subseteq B_\eta(y)$. For $g \in G$ now define

$$A(j, g) := \beta\left(g^{-1}, A_j^{\beta(g,y)}\right) .$$

Clearly we have $\nu\left(A(j, g)\right) > \kappa$. Note that if $E_g \cap A(j, g) = \emptyset$ then we calculate $\nu(E_g) + \nu(A(j, g)) \leqslant \nu(B_\eta(y))$ as

$$E_g \cup A(j, g) \subseteq \beta\left(g^{-1}, B_\eta(\beta(g, y))\right) = B_\eta(y) .$$

This inequality is violated whenever

$$\nu(E_g) > \nu(B_\eta(y)) - \frac{\kappa}{2} \tag{23}$$

This logic is independent of the choice of $j \in \{1, 2\}$. So whenever equation (23) holds E_g intersects both $A(1, g)$ and $A(2, g)$. However

$$d_X(\rho(\beta(g, A(1, g))), \rho(\beta(g, A(2, g)))) > \frac{\varepsilon}{4}.$$

We use the definition of E_g to translate from ρ to φ. We know that whenever (23) holds there are $y_j^g \in A(j, g) \cap E_g$. On the one hand, we know that

$$d_X\left(\alpha(g, \varphi(y_j^g)), \alpha(g, \rho(y_j^g))\right) \leqslant \frac{\varepsilon}{8} \tag{24}$$

by the definition of E_g. On the other hand, since $y_j^g \in A(j, g)$ we know that $\beta(g, y_j^g) \in B_\eta(\beta(g, y))$ and $d_X(\rho(\beta(g, y_1^g)), \rho(\beta(g, y_2^g))) > \frac{\varepsilon}{4}$. As φ and ρ are left inverses of the factor map π they are themself equivariant. So we conclude that

$$d_X\left(\alpha(g, \rho(y_1^g)), \alpha(g, \rho(y_2^g))\right) > \frac{\varepsilon}{4}. \tag{25}$$

Combining (24) and (25) we obtain by the inverse triangle inequality that

$$d_X\left(\alpha(g, \varphi(y_1^g, g)), \alpha(g, \varphi(y_2^g, g))\right) > \frac{\varepsilon}{8}. \tag{26}$$

Further remember that $\varphi : B_\eta(y) \to B_\delta(x)$ and thus $\varphi(y_j^g) \in B_\delta(x)$. So whenever (23) holds, we have $\mathsf{diam}(\alpha(g, B_\delta(x))) > \frac{\varepsilon}{4}$. However as we have seen in (22) this happens for a subset $H \subseteq G$ with $D_{\mathcal{F}}(H) = 1$. As δ was arbitrary we conclude that x is not $\mathcal{F}$-frequently stable. Further, as x was arbitrary we conclude that no point is $\mathcal{F}$-frequently stable. □

7 Diam-Mean Equicontinuity - Almost surely 1-1 MEF

In this section we need to assume **unimodularity** of the group:

Definition 7.0.1 (Unimodularity)**.** A group G is called unimodular if and only if a right HAAR measure is as well a left HAAR measure.

Remark 7.0.1.1. i) Obviously all Abelian groups are unimodular.

ii) As on discrete groups the Haar measure is the counting measure card we easily see that all discrete groups are unimodular.

iii) It is also well known that all compact groups are unimodular, the argument here is elegant but a bit more involved: Let m be any left Haar measure. Now for any $g \in G$ the measure $n : A \mapsto m(Ag)$ is a left Haar measure as well. By uniqueness of the Haar measure, there must be $c_g > 0$ such that $n = c_g m$. Now $\Delta : g \mapsto c_g$ forms a continuous group homomorphism $\Delta : G \to (\mathbb{R}^+, \cdot)$. As G is assumed to be compact this continuous map must be bounded. A group homomorphism to the multiplicative group of real numbes can only be bounded if it is constant to 1, so $\Delta \equiv 1$. This shows that m is right invariant as well and G is unimodular.

The space X is compact metrizable and d_X is a metric on X. The group G shall be locally compact and unimodular. Let m denote a (**right** and **left**) Haar measure on G and fix a **right** and **left** Følner sequence $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$. This assumption is necessary as we need the Lindenstrauss Ergodic Theorem 1.12.16 which only holds for (tempered) left Følner sequences but also need to use right invariance.

Fix $(X, G, \alpha) \in \mathsf{CMetDyn}(G)$ and denote $\mathbf{X} := (X, G, \alpha)$.

7.1 Diam-Mean Equicontinuity

We follow closely the presentation given in [García-Ramos et al., 2021].

Definition 7.1.1 (Diam-Mean Equicontinuity)**.** We call $x \in X$ a **$\mathcal{F}$-diam-mean equicontinuity point** if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha\left(g, B_\delta(x)\right)\right) \, \mathrm{d}m(g) < \varepsilon\,. \tag{27}$$

The system $\mathbf{X}$ is called **$\mathcal{F}$-diam-mean equicontinuous** whenever every $x \in X$ is an $\mathcal{F}$-diam-mean equicontinuity point.

The compactness of X allows to deduce the uniform version of diam-mean equicontinuity from the pointwise:

Lemma 7.1.2. $\mathbf{X}$ *is $\mathcal{F}$-diam-mean equicontinuous if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that (27) holds for all $x \in X$.*

Proof. It is clear that the latter implies the former. Now let $\mathbf{X}$ be $\mathcal{F}$-diam-mean equicontinuous. Let $\varepsilon > 0$. For $x \in X$ the set $D_x := \{\delta_x > 0 \mid (27) \text{ holds}\}$ is non-empty. Let $\delta(x) := \sup D_x$. It is sufficient to show that $\delta : X \to \mathbb{R}^+$ has a minimum. Further, it suffices to prove that δ is lower semi-continuous. We will thus show that for $x_0 \in X$ and any $\eta > 0$ we have that

$$B_\eta(x_0) \subseteq \{x \in X \mid \delta(x) \geqslant \delta(x_0) - \eta\}\ .$$

Let $\delta' > 0$ be such that (27) holds for x_0, i.e.

$$\limsup_{N\to\infty} \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha\left(g, B_{\delta'}(x_0)\right)\right) \, \mathrm{d}m(g) < \varepsilon\,.$$

Pick any $\eta : 0 < \eta < \delta'$ and $x \in B_\eta(x_0)$. Then $B_{\delta'-\eta}(x) \subseteq B_{\delta'}(x_0)$. In particular

$$\mathsf{diam}\left(\alpha\left(g, B_{\delta'-\eta}(x)\right)\right) \leqslant \mathsf{diam}\left(\alpha\left(g, B_{\delta'}(x_0)\right)\right)\ .$$

Thus $\delta(x) \geqslant \delta' - \eta$. By taking the supremum over δ' we conclude $\delta(x) \geqslant \delta(x_0) - \eta$. $\square$

Again compactness allows to further strengthen diam-mean equicontinuity:

Definition 7.1.3 (ε-Stability in the Mean)**.** Let $U \subseteq X$. The subset U is called **$\mathcal{F}$-ε-stable in the mean** if and only if

$$\sup \left\{ \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha(g, U)\right) \, dm(g) \,\middle|\, n \in \mathbb{N} \right\} < \varepsilon \, .$$

Lemma 7.1.4. **X** *is $\mathcal{F}$-diam-mean equicontinuous if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in X$ the ball $B_\delta(x)$ is $\mathcal{F}$-ε-stable in the mean.*

Proof. It is trivial that the latter implies the former. Conversely, let **X** be $\mathcal{F}$-diam mean equicontinuous. By Lemma 7.1.2 we know that for any $\varepsilon > 0$ there is an $\delta > 0$ such that (27) holds for any $x \in X$. Assume that there is no δ' such that for any $x \in X$ the set $B'_\delta(x)$ is $\mathcal{F}$-ε-stable. Then there is $x_n \in X$ and $N_n \in \mathbb{N}$ such that

$$\frac{1}{m(F_{N_n})} \int_{F_{N_n}} \mathsf{diam}\left(\alpha(g, B_{\frac{1}{n}}(x_n))\right) \, dm(g) \geqslant \varepsilon \, .$$

Recall that F_k is compact for any $k \in \mathbb{N}$. Observe that the family

$$\left\{ \alpha\left(g, \cdot\right) \,\middle|\, g \in \bigcup_{n=1}^{K} F_{N_n} \right\}$$

is equicontinuous for any $K \in \mathbb{N}$. So for any $K \in \mathbb{N}$ there is $\delta > 0$ such that

$$\max \left\{ \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) \, dm(g) \,\middle|\, n < K \right\} < \varepsilon$$

for all $x \in X$. We conclude that $N_n \xrightarrow{n\to\infty} \infty$.

X is sequentially compact. W.l.o.g. x_n is convergent with limit $x \in X$. For any $\eta : 0 < \eta < \delta$ there is $N \in \mathbb{N}$ such that for $n > N$ we have $\eta + \frac{1}{n} < \delta$ and $d_X(x_n, x) < \eta$. So $B_{\frac{1}{n}}(x_n) \subseteq B_\delta(x)$ for $n > N$ and further

$$\begin{aligned} \varepsilon &\leqslant \frac{1}{m\left(F_{N_n}\right)} \int_{F_{N_n}} \mathsf{diam}\left(\alpha(g, B_{\frac{1}{n}}(x_n))\right) \, dm(g) \\ &\leqslant \frac{1}{m\left(F_{N_n}\right)} \int_{F_{N_n}} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) \, dm(g) \, . \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \frac{1}{m\left(F_n\right)} \int_{F_n} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) \, dm(g) \geqslant \varepsilon$$

in contradiction to (27). □

Lemma 7.1.5. *For any $x \in X$ it is equivalent:*

(A) *x is an $\mathcal{F}$-diam-mean equicontinuity point.*

(B) *For every $\eta > 0$ there is $\delta > 0$ such that*

$$\bar{D}_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right) < \eta \, .$$

Proof. We assume w.l.o.g. that $\mathsf{diam}(X) = 1$.

(A) implies (B): We proceed by contraposition. Assume that there is an $\eta > 0$ such that for any $\delta > 0$ we have

$$\bar{D}_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right) \geqslant \eta\,.$$

We define $\varepsilon := \eta^2$. For any $\delta > 0$ we have

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g)\\ &\geqslant \eta \cdot \bar{D}_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right)\\ &\geqslant \eta \cdot \eta = \varepsilon\,.\end{aligned}$$

This shows that x is no $\mathcal{F}$-diam-mean equicontinuity point.

(B) implies (A): Assume (B) holds. For any $\varepsilon > 0$ let $\eta := \frac{\varepsilon}{2}$. There is $\delta > 0$ such that

$$\bar{D}_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right) < \eta\,.$$

Let $H := \{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}$ and calculate

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g)\\ &\leqslant \mathsf{diam}(X) \cdot \bar{D}_{\mathcal{F}}(H) + \eta \cdot \bar{D}_{\mathcal{F}}(H^c)\\ &\leqslant 1 \cdot \eta + \eta \cdot 1 = 2\eta = \varepsilon\,.\end{aligned}$$

As ε was arbitrary x is an $\mathcal{F}$-diam-mean equicontinuity point. □

7.2 Banach Diam-Mean Equicontinuity

Definition 7.2.1 (Banach diam-mean Equicontinuity)**.** A point $x \in X$ is called a **Banach $\mathcal{F}$-diam-mean equicontinuity point** if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\limsup_{n\to\infty} \sup_{h\in G} \frac{1}{m(F_n)} \int_{F_n\cdot h} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g) < \varepsilon\,. \tag{28}$$

The system $\mathbf{X}$ is called **Banach $\mathcal{F}$-diam-mean equicontinuous** whenever every $x \in X$ is a Banach $\mathcal{F}$-diam-mean equicontinuity point.

Similarly to Lemma 7.1.2 we can prove by compactness

Lemma 7.2.2. *The system $\mathbf{X}$ is Banach $\mathcal{F}$-diam-mean equicontinuous if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that (28) holds for all $x \in X$.*

Analogously to Lemma 7.1.5 compactness yields

Lemma 7.2.3. *For any $x \in X$ it is equivalent:*

(A') *x is a Banach $\mathcal{F}$-diam-mean equicontinuity point.*

(B') *For every $\eta > 0$ there is $\delta > 0$ such that*

$$BD^*_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right) < \eta\,.$$

Proof. The proof is analogous to the one of Lemma 7.1.5. It is only provided for the sake of completeness. We assume w.l.o.g. that $\mathsf{diam}(X) = 1$.

(A') implies (B'): We proceed by contraposition. Assume that there is an $\eta > 0$ such that for any $\delta > 0$ we have

$$BD^*_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right) \geqslant \eta\,.$$

We define $\varepsilon := \eta^2$. For any $\delta > 0$ we have

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{h\in G} \frac{1}{m(F_n)} \int_{F_n \cdot h} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g)\\ &\geqslant \eta \cdot BD^*_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right)\\ &\geqslant \eta \cdot \eta = \varepsilon\,.\end{aligned}$$

This shows that x is no Banach $\mathcal{F}$-diam-mean equicontinuity point.

(B') implies (A'): Assume (B') holds. For any $\varepsilon > 0$ let $\eta := \frac{\varepsilon}{2}$. There is $\delta > 0$ such that

$$BD^*_{\mathcal{F}}\left(\{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}\right) < \eta\,.$$

Let $H := \{g \in G \mid \mathsf{diam}\left(\alpha(g, B_\delta(x))\right) > \eta\}$ and calculate

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{h\in G} \frac{1}{m(F_n)} \int_{F_n \cdot h} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g)\\ &\leqslant \mathsf{diam}(X) \cdot BD^*_{\mathcal{F}}(H) + \eta \cdot BD^*_{\mathcal{F}}(H^c)\\ &\leqslant 1 \cdot \eta + \eta \cdot 1 = 2\eta = \varepsilon\,.\end{aligned}$$

As ε was arbitrary x is an $\mathcal{F}$-diam-mean equicontinuity point. □

7.3 Main Theorem

Recall the standing assumptions that X is compact metric and G is a locally compact, right amenable group, m is a right HAAR measure on G and $\mathcal{F}$ a right Følner sequence.

Theorem 7.3.1. *Suppose that G is σ-compact, $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ is tempered and $(X, G, \alpha) = \mathbf{X} \in \mathsf{CMetDyn}(G)$ is minimal. Let $(\mathbf{Y}, \pi)$ be a* `mef` *of $\mathbf{X}$. Then the following are equivalent:*

(i) $\pi : \mathbf{X} \to \mathbf{Y}$ *is almost surely 1-1.*

(ii) $\mathbf{X}$ *is Banach $\mathcal{F}$-diam-mean equicontinuous.*

(iii) $\mathbf{X}$ *is $\mathcal{F}$-diam-mean equicontinuous.*[90]

[90] This is a straightforward generalization of the proof and statement of Theorem 4.12 in [García-Ramos et al., 2021].

Remark 7.3.1.1. Note that the invertibility property *(i)* is independent of the tempered Følner sequence $\mathcal{F}$. Thus diam-mean equicontinuity with respect to one tempered Følner sequence is equivalent to diam-mean equicontinuity with respect to all tempered Følner sequences.

The proof of the implication *(i)* to *(ii)* is again quite technical. We thus present its rough strategy first.

Strategy. Let $\varepsilon > 0$. The set $\mathcal{I}$ of points $y \in Y$ with exactly one preimage has full measure. On $\mathcal{I}$ we can define an inverse $x : \mathcal{I} \to X$ of π. By regularity we can find a compact subset $K \subseteq \mathcal{I}$ with $\nu(K) > 1 - \varepsilon$. Now for $y \in K$ we can find a $\delta_y > 0$ such that $\pi^{-1}(B_{2\delta_y}(y)) \subseteq B_\varepsilon(x(y))$. Cover K by finitely many $B_{\delta_{y_i}}(y_i)$ and let K_ε be the union of those. We can find $\eta > 0$ such that for all $y \in K$ there is an i such that $B_\eta(y) \subseteq B_{\delta_{y_i}}(y_i)$. By continuity there is $\delta > 0$ such that $\mathsf{diam}(\pi(B_\delta(x))) < \frac{\eta}{2}$. Note that for any $y \in Y$ the set H of $g \in G$ such that $\beta(g, y) \in K_\varepsilon$ has density greater than $1 - \varepsilon$ by the Uniform Ergodic Theorem 1.13.5. But from the above we can show that for $g \in H$ we have $\alpha(g, B_\delta(x)) \subseteq B_\varepsilon(x(y_i))$ for some i. So for $g \in H$ we have $\mathsf{diam}(\alpha(g, B_\delta(x))) < 2\varepsilon$ which implies the diam-mean equicontinuity. □

Proof. It is clear that *(ii)* implies *(iii)* as

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{h\in G} \frac{1}{m(F_n)} \int_{F_n\cdot h} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g) \\
\geqslant &\limsup_{n\to\infty} \frac{1}{m(F_n)} \int_{F_n} \mathsf{diam}\left(\alpha(g, B_\delta(x))\right)\, \mathrm{d}m(g)\,.
\end{aligned}$$

We simplify notation: Let $(Y, G, \beta) = \mathbf{Y}$ where β is w.l.o.g. assumed to be an isometry. By d_X and d_Y we denote the metrics on X and Y respectively. Further let ν be the unique invariant probability on the minimal and equicontinuous system $\mathbf{Y}$.

***(i)* implies *(ii)*:** Define

$$\mathcal{I} := \left\{ y \in Y \;\middle|\; \mathsf{card}\left(\pi^{-1}\left(\{y\}\right)\right) = 1 \right\} .$$

As for $y \in \mathcal{I}$ there is a unique preimage we can define

$$x : \mathcal{I} \longrightarrow X,\ y \longmapsto x(y) \in \pi^{-1}\left(\{y\}\right) .$$

Now as π is almost surely 1-1 we know that $\nu(\mathcal{I}) = 1$.

Let $\varepsilon > 0$. Note that ν is regular by Ulam's Theorem 1.8.9. There is a compact $K \subseteq \mathcal{I}$ such that $\nu(K) > 1 - \varepsilon$. We reuse an argument from the proof of Lemma 3.1.4. Recall that the multi-valued map

$$D_\cdot(y) : \mathbb{R}_0^+ \twoheadrightarrow Y,\ \eta \longmapsto \{y' \in Y \mid d_Y(y, y') \leqslant \rho\} \;.$$

has a closed graph for any $y \in Y$ by continuity of d_Y. We conclude with Theorem 1.5.9 that $D_\cdot(y)$ is upper hemi-continuous. So $\pi^{-1} \circ D_\rho$ is upper hemi-continuous. Thus $E := \{\eta \geqslant 0 \mid \pi^{-1}\left(D_\eta(y)\right) \subseteq B_\varepsilon(x(y))\}$ is open. Now let $y \in K$. Note that $0 \in E$ as $y \in K$ is has exactly one preimage. Thus we know that there is $\eta_y \in E$ such that $\eta_y > 0$. Let $\delta_y := \frac{\eta_y}{2}$. We conclude that for all $y \in K$ there is $\delta_y > 0$ such that

$$\pi^{-1}\left(B_{2\delta_y}(y)\right) \subseteq B_\varepsilon(x(y)).$$

Clearly $\left\{B_{\delta_y}(y) \mid y \in K\right\}$ is an open cover of K. By compactness of K there is $n \in \mathbb{N}$ and $\{y_1, \ldots, y_n\} \subseteq K$ such that $\bigcup_{i=1}^n B_{\delta_{y_i}}(y_i) \subseteq K$. We define

$$\eta := \min\{\delta_{y_i} \mid i \in \{1, \ldots, n\}\} \qquad \text{and} \qquad K_\varepsilon := \bigcup_{i=1}^n B_{\delta_{y_i}}(y_i).$$

Trivially K_ε is open with $\nu(K_\varepsilon) > 1-\varepsilon$. By definition we have for any $y \in K_\varepsilon$ that there is $i \in \{1, \ldots, n\}$ such that $y \in B_{\delta_{y_i}}(y_i)$. We conclude

$$B_\eta(y) \subseteq B_{\delta_{y_i}}(y) \subseteq B_{2\delta_{y_i}}(y_i).$$

By continuity there is $\delta > 0$ such that if $\mathsf{diam}(U) < 2\delta$ then $\mathsf{diam}(\pi(U)) < \frac{\eta}{2}$. Choose any $x \in X$ and let $U := B_\delta(x)$. Note that $\mathsf{diam}(U) < 2\delta$. As β is isometric we know that for all $g \in G$ we have

$$\mathsf{diam}\left(\beta(g, \pi(U))\right) < \frac{\eta}{2}.$$

Let $\beta(\cdot, y)^{-1}(K_\varepsilon) := \{g \in G \mid \beta(g, y) \in K_\varepsilon\}$. For $g \in \beta(\cdot, y)^{-1}(K_\varepsilon)$ we have that

$$\exists i \in \{1, \ldots, n\} : \beta(g, \pi(U)) \subseteq B_\eta\left(\beta(g, y)\right) \subseteq B_{2\delta_{y_i}}(y_i).$$

Which implies that

$$\begin{aligned} \alpha(g, U) &\subseteq \pi^{-1}\left(\pi\left(\alpha(g, U)\right)\right) \\ &\subseteq \pi^{-1}\left(\beta\left(g, \pi(U)\right)\right) \\ &\subseteq \pi^{-1}\left(B_{2\delta_{y_i}}(y_i)\right) \\ &\subseteq B_\varepsilon(x(y_i)) \end{aligned} \tag{29}$$

Note that K_ε is open and K is closed. So by URYSOHN's Lemma 1.7.2 there is a continuous function $f : X \to [0,1]$ such that $f|_{K_\varepsilon^c} = 0$ and $f|_K = 1$. By Corollary 4.4.1.1 $\mathbf{Y}$ is uniquely ergodic. By Theorem 1.13.5 the ergodic averages of f converge uniformly to $\int f \,\mathrm{d}\nu \geqslant \nu(K) > 1-\varepsilon$. In particular there is $N \in \mathbb{N}$ such that for all $n > N$ and any $y \in Y$ we have

$$\frac{1}{m(F_n)} \int_{F_n} \mathbb{1}_{K_\varepsilon} \circ \beta(g, y) \,\mathrm{d}m(g) \geqslant \frac{1}{m(F_n)} \int_{F_n} f \circ \beta(g, y) \,\mathrm{d}m(g) \geqslant 1 - \varepsilon.$$

As we have that $\beta(h, y) \in Y$ for any $y \in Y$ and $h \in G$ and as $\beta(g, \beta(h, y)) = \beta(g \cdot h, y)$ we conclude that for any $n > N$, $y \in Y$ and $h \in G$ it holds that

$$\frac{1}{m(F_n)} \int_{F_n \cdot h} \mathbb{1}_{K_\varepsilon} \circ \beta(g, y) \,\mathrm{d}m(g) \geqslant 1 - \varepsilon.$$ [91]

We conclude that

$$BD^*_{\mathcal{F}}\left(\beta(\cdot,y)^{-1}(K_\varepsilon)^c\right) < \varepsilon\,. \tag{30}$$

Now from (29) we learn that

$$\beta(\cdot,y)^{-1}(K_\varepsilon) \subseteq \{g \in G \mid \mathsf{diam}(\alpha(g,U)) \leqslant 2\varepsilon\}\,. \tag{31}$$

Combining (30) and (31) we can conclude using Lemma 7.2.3 that **X** is Banach $\mathcal{F}$-diam-mean equicontinuous.

***(iii)* implies *(i)*:** We proceed by contradiction. As **X** is $\mathcal{F}$-diam-mean equicontinuous it is in particular $\mathcal{F}$-mean equicontinuous and frequently stable. As **X** is minimal and $\mathcal{F}$-mean equicontinuous it is uniquely ergodic by Proposition 5.1.7. By Theorem 6.0.2 we know that π is almost 1-1 (generically 1-1).

Assume that π is not almost surely 1-1. Recall the definition

$$\pi_*\mathsf{diam}(y) := \mathsf{diam}\left(\pi^{-1}\left(\{y\}\right)\right)$$

and conclude

$$\nu\left(\{y \in Y \mid \pi_*\mathsf{diam}(y) > 0\}\right) > 0\,.$$

By the Archimedean property of $\mathbb{R}$ this means that there is $\varepsilon > 0$ such that

$$2\eta := \nu\Big(\underbrace{\{y \in Y \mid \pi_*\mathsf{diam}(y) \geqslant \varepsilon\}}_{:=A_\varepsilon}\Big) > 0\,.$$

By the Lindenstrauss Ergodic Theorem 1.12.16 there is $y \in A_\varepsilon$ and $N \in \mathbb{N}$ such that for $k > N$

$$\frac{m\left(\{g \in G \mid \beta(g,y) \in A_\varepsilon\} \cap F_k\right)}{m\left(F_k\right)} = \frac{1}{m\left(F_k\right)}\int_{F_k} \mathbb{1}_{A_\varepsilon} \circ \beta(g,y)\,\mathrm{d}m(g) > \eta\,. \tag{32}$$

Recall that π is almost 1-1 (generically 1-1). In particular, the set of points with exactly one preimage is dense in Y. Using Lemma 3.1.3 we conclude that for any non-empty open set $U \subseteq X$ its image $\pi(U)$ has non-empty interior. There is thus a point with exactly one preimage $y_U \in \mathrm{int}(\pi(U)) =: V$.

As π^{-1} is upper hemi-continuous and U is open the set

$$\left\{y \in Y \;\middle|\; \pi^{-1}\left(\{y\}\right) \subseteq U\right\} =: W$$

is open. As $\pi^{-1}(y_U) \subseteq U$ we conclude that there is $\delta > 0$ such that $B_\delta(y_U) \subseteq W$. By minimality the orbit of y is dense. In particular there is $g_U \in G$ such

[91] This is the point where we require that the **rightness** of the Følner sequence. Here we can easily add the multiplication from the right by the above argument. However this argument can not be used to add an arbitrary factor from the left.

that $\beta(g_U, y) \in B_\delta(y_U)$. This yields $U \supseteq \pi^{-1}(\beta(g_U, y)) = \alpha(g_U, \pi^{-1}(\{y\}))$ and further by monotonicity

$$\alpha(g, U) \supseteq \alpha\left(g, \alpha\left(g_U, \pi^{-1}(\{y\})\right)\right) . \tag{33}$$

Equation (33) together with the definition of A_ε implies that

$$\{g \in G \mid \mathsf{diam}(\alpha(g, U)) > \varepsilon\} \supseteq \{g \in G \mid \beta(g \cdot g_U, y) \in A_\varepsilon\} := H$$

However by the choice of y

$$\bar{D}_{\mathcal{F}}(H) = \bar{D}_{\mathcal{F}}(H \cdot g_U) = \nu(A_\varepsilon) > \eta . \tag{34}$$

As η is independent of U (34) holds in particular for any $U = B_\delta(x)$. Using Lemma 7.1.5 we obtain a contradiction to the $\mathcal{F}$-diam-mean equicontinuity of $\mathbf{X}$. □

Note that the proof of *"(i) implies (ii)"* does neither assume nor use the temperedness of the Følner sequences $\mathcal{G} \in \mathcal{L}$. We see that the existence of one tempered Følner sequence $\mathcal{F}$ such that $\mathbf{X}$ is $\mathcal{F}$-diam-mean equicontinuous implies that $\mathbf{X}$ is $\mathcal{G}$-diam-mean equicontinuous for any Følner sequence $\mathcal{G}$.

Note that the left hand side of (27) only gets smaller when going to a subsequence. So diam-mean equicontinuity is preserved by going to a subsequence. Any Følner sequence however has a tempered subsequence by Proposition 1.4 in [Lindenstrauss, 2001].

So if there is a Følner sequence $\mathcal{F}$ such that $\mathbf{X}$ is $\mathcal{F}$-diam-mean equicontinuous then there is a tempered Følner sequence $\mathcal{F}'$ such that $\mathbf{X}$ is $\mathcal{F}'$-diam-mean equicontinuous. Therefore, $\mathbf{X}$ is $\mathcal{G}$-diam-mean equicontinuous for any Følner sequence $\mathcal{G}$.

Thus we obtain:

Theorem 7.3.2. *Let $\mathbf{X} = (X, G, \alpha) \in \mathsf{CMetDyn}(G)$ satisfy the standing assumptions that G is locally compact and equipped with a left Følner sequence $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$. Let X be compact and $\mathbf{X}$ be minimal. Suppose that $(\mathbf{Y}, \pi)$ is a* `mef` *of $\mathbf{X}$. The following are equivalent:*

(i) $\pi : \mathbf{X} \to \mathbf{Y}$ *is almost surely 1-1.*

(ii) $\mathbf{X}$ *is Banach $\mathcal{F}$-diam-mean equicontinuous.*

(iii) $\mathbf{X}$ *is $\mathcal{F}$-diam-mean equicontinuous.*

8 Discrete Spectrum of Measurable Eigenfunctions

Let G be Abelian. All the factor maps to a `mef` we considered where at least topo-isomorphies. Assume that $\mathbf{X} \in \mathsf{CHausDyn}(G)$ is uniquely ergodic and that

$$\hat{\mathfrak{B}}\mathbf{X} \cong \hat{\mathfrak{B}}\left(\mathsf{MEF}(\mathbf{X})\right) .$$

Recall that $\mathsf{MEF}(\mathbf{X})$ is conjugate to a dynamical system on a group compactification (H, ψ) where the dynamic is given by the rotation

$$\beta(h, g) = \psi(g) \cdot h \, .$$

Thus, $\hat{\mathfrak{B}}\mathbf{X}$ is isomorphic to the measure preserving dynamical system of a rotation on a compact topological group equipped with the BOREL-σ-algebra and the (left) HAAR measure.

Those measure preserving systems isomorphic to such group rotations are exactly the ones with discrete spectrum.

We follow closely the presentation given in [Hermle and Kreidler, 2022].

Definition 8.0.1 (KOOPMAN Representation)**.** Fix $(Z, G, \mu, \alpha) \in \mathsf{ProbDyn}(G)$ and denote $\mathbf{Z} := (Z, G, \mu, \alpha)$. The associated bounded and strongly continuous group presentation on the Banach space $L^1(\mu)$ is given by

$$M_{\mathbf{Z}}(f) = f \circ \alpha(g, \cdot) \, .$$

We call $M_{\mathbf{Z}}$ the **Koopman representation** of the `mpds` $\mathbf{Z}$.

Definition 8.0.2 (`dsme`)**.** Let $\mathbf{Z} \in \mathsf{ProbDyn}(G)$. We say that $\mathbf{Z}$ has **discrete spectrum (with measurable eigenfunctions)** (`dsme`) if and only if $M_{\mathbf{Z}}$ has discrete spectrum as in Definition 4.5.2.

We will write $\sigma_{\mathsf{p}}(\mathbf{Z}) := \sigma_{\mathsf{p}}(M_{\mathbf{Z}})$ where the latter is defined by Definition 4.5.5.

Proposition 8.0.3. *Let G be Abelian. If $\mathbf{Z}$ is ergodic, then $\sigma_{\mathsf{p}}(\mathbf{Z}) \leq G^*$, i.e. the spectrum is a subgroup.*[92]

Now in [Hermle and Kreidler, 2022, p.20] the well-known HALMOS-VON NEUMANN THEOREM for `mpds` is presented:

Theorem 8.0.4. *Let G be Abelian.*

(i) Let $\mathbf{X}, \mathbf{Y} \in \mathsf{ProbDyn}(G)$ be ergodic and have `dsme`. Then we have $\mathbf{X} \cong \mathbf{Y}$ if and only if $\sigma_{\mathsf{p}}(\mathbf{X}) = \sigma_{\mathsf{p}}(\mathbf{Y})$.

(ii) For every subgroup $\sigma \leq G^*$ there is an ergodic system $\mathbf{X} \in \mathsf{ProbDyn}(G)$ with `dsme` such that $\sigma_p(\mathbf{X}) = \sigma$.

(iii) Let $\mathbf{X} \in \mathsf{ProbDyn}(G)$ be ergodic and have `dsme`. There is a group compactification (H, ψ) of G^* such that $\mathbf{X} \cong \mathbf{H}$ where $\mathbf{H} = (H, G, m_H, \beta)$ with m_H being the left HAAR measure on H and

$$\beta(g, h) = \psi(g) \cdot h \, .$$

[92] This is Proposition 4.3 in [Hermle and Kreidler, 2022, p.19].

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Nomenclature

CHaus Full subcategory CHaus $\subset$ Top of all compact Hausdorff spaces, . . page 8

CHausDyn(G) Category of compact Hausdorff topological dynamical systems of a group G with factor maps as morphisms, ... page 51

CMetDyn(G) Category of compact metric topological dynamical systems of a group G with factor maps as morphisms, ... page 51

EquiDyn(G) Category of equicontinuous dynamical systems with factor maps as morphisms, ... page 52

Grp Category of groups with group homomorphisms as morphisms, ... page 45

Meas Category of all measure spaces with measurable measure-preserving maps as morphisms, ... page 9

PreMeas Category of all measurable spaces with measurable maps as morphisms., page 33

Prob Full subcategory Prob $\subset$ Meas of all probability spaces, ... page 9

Set Category of all sets with functions as morphisms, ... page 8

Top Category of all topological spaces with continuous maps as morphisms, page 8

TopDyn(G) Category of topological dynamical systems of a group G with factor maps as morphisms, ... page 51

TopGrp Category of topological groups with continuous group homomorphisms as morphisms, ... page 45

ProbDyn(G) Category of all measure preserving dynamical systems on probability spaces, ... page 54

Comp(G) Category of all group compactifications of G, ... page 53

measCHausDyn(G) Category of all topological dynamical systems equipped with a fixed invariant probability, ... page 76

pt The subscript pt denotes that a category of dynamical system is pointed, page 53

ueCHausDyn(G) Full Subcategory of all uniquely ergodic topological dynamical systems, ... page 75

$\mathfrak{B}$ Functor sending a topological space to the measurable space with the Borel-σ-algebra, ... page 33

MEF Functor MEF : CHausDyn$(G) \to$ EquiDyn(G) mapping a topological dynamical system to its unique maximal equicontinuous factor, ... page 68

$\hat{\mathfrak{B}}$ Functor sending an uniquely ergodic topological dynamical system to the measure preserving dynamical system equipped with the BOREL-σ-algebra and the unique ergodic BOREL probability, ... page 75

$B \leq A$ B is an algebraic substructure of A, ... page 7

$B_\varepsilon(x)$ The open ball of radius ε around x, ... page 25

G^* The Pontryagin dual of a topological group G, ... page 47

$L^p(\mu)$ Banach space of all functions f such that $\int |f|^p \, \mathrm{d}\mu < \infty$, ... page 29

Y^X The set of functions from X to Y, ... page 7

$\mathfrak{A} * \mathfrak{F}$ If $\mathfrak{A}$ and $\mathfrak{F}$ are systems of sets then $\mathfrak{A} * \mathfrak{F}$ denotes the system of boxes, i.e. $\mathfrak{A} * \mathfrak{F} := \{A \times F \mid A \in \mathfrak{A}, F \in \mathfrak{F}\}$., ... page 27

$\mathfrak{A} \otimes \mathfrak{F}$ If $\mathfrak{A}$ and $\mathfrak{F}$ are σ-algebras then $\mathfrak{A} \otimes \mathfrak{F}$ denotes the product σ-algebra., page 27

$\mathrm{Aut}(\mathbf{X})$ Group of equivariant automorphisms of $\mathbf{X}$, ... page 63

$\mathrm{Aut}_{\mathsf{Top}}(\mathbf{X})$ Topological group of equivariant automorphisms of $\mathbf{X}$ equipped with the topology of pointwise convergence, ... page 82

$\mathcal{C}(X, Y)$ The set of continuous functions from X to Y, ... page 32

$\mathcal{C}_c(X, \mathbb{R})$ The set of continuous functions from X to $\mathbb{R}$ with compact support, page 32

$\mathrm{End}(X)$ Group of self-homeomorphisms of a topological space X, ... page 63

$\mathrm{End}(X)$ Monoid of endomorphisms of a topological space X, ... page 63

$\mathrm{End}(\mathbf{X})$ Monoid of equivariant endomorphisms of $\mathbf{X}$, ... page 63

$\mathcal{F}(X, Y)$ The set of functions from X to Y, ... page 7

$\mathfrak{F} \leq \mathfrak{A}$ $\mathfrak{F}$ is a sub-σ-algebra of $\mathfrak{A}$, ... page 7

$\mathrm{Hom}_{\mathsf{C}}(A, B)$ The morphisms between objects A and B from a category C, . page 8

$\mathcal{M}(X, Y)$ The set of measurable functions from X to Y, ... page 27

$\mathfrak{P}(X)$ The power set of X, ... page 7

$\mathcal{U}(x)$ The neighbourhood filter of x, ... page 32

$\mathrm{dom}(f)$ The domain of the function f, ... page 7

$\mathrm{int}(A)$ The interior of A, ... page 32

proj_A The projection onto A, ... page 7

$\sigma(\mathfrak{E})$ The σ-algebra generated by $\mathfrak{E}$, ... page 27

$f : X \twoheadrightarrow Y$ A multivalued function from X to Y, ... page 24

$f_*\mu$ The pushforward measure $f_*\mu(A) = \mu(f^{-1}(A))$, ... page 28

Borel-σ-algebra The σ-algebra generated by the open sets of a given topological space, ... page 32

G^{op} The opposite (topological) group where the order of multiplication is reversed, page 45

$\mathbf{X}^{\mathsf{op}}$ The opposite `tds`, ... page 51

m^{op} The opposite measure of a given Haar measure m defined by $m^{\mathsf{op}}(A) = m(A^{-1})$, ... page 46

`icer` Shorthand for invariant closed equivalence relation, ... page 66

`mef` Shorthand for "maximal equicontinuous factor", ... page 62

`tds` Shorthand for topological dynamical system, ... page 52

$BD^*_{\mathcal{F}}(A)$ The upper Banach density of $A \subset G$ with respect to the ergodic net $\mathcal{F}$, page 50

$BD_{\mathcal{F}}(A)$ The Banach density of $A \subset G$ with respect to the ergodic net $\mathcal{F}$, page 50

$C_{\mathbf{X}}$ The Koopman representation of the `tds` $\mathbf{X}$ on the space of continuous functions $\mathcal{C}(X, \mathbb{C})$, ... page 91

$D_{\mathcal{F}}(A)$ The density of $A \subset G$ with respect to the ergodic net $\mathcal{F}$, ... page 48

$\mathfrak{A}_\mu$ The completion of the σ-algebra $\mathfrak{A}$ w.r.t. the measure μ, ... page 36

$\mathbb{E}_\mu [T \mid \mathfrak{F}]$ Conditional Expectation of T w.r.t. to the sub-σ-algebra $\mathfrak{F}$ and the measure μ, ... page 31

$\bar{D}_{\mathcal{F}}(A)$ The upper density of $A \subset G$ with respect to the ergodic net $\mathcal{F}$, ... page 48

δ_x The Dirac measure in x, ... page 28

$\underline{D}^{\mathcal{F}}(A)$ The lower density of $A \subset G$ with respect to the ergodic net $\mathcal{F}$, ... page 48

$BD^{\mathcal{F}}_*(A)$ The lower Banach density of $A \subset G$ with respect to the ergodic net $\mathcal{F}$, page 50