Topology

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Based on Notes by Tobias Jäger

Contents

Disclaimer

This is work in progress and will develop during the running semester!
In the likely case of typos or other errors please inform me, so that I can correct them.

0.1 Some Motivation and Introduction

The theory of metric spaces can be applied very broadly. Topology is a generalization of this theory. Why would one generalize such a general theory even further?
This can be motivated by examples where the theory of metric spaces does not apply!

Pointwise Convergence: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ a function. We say that f_n **converges to** f **pointwise** if

$$\forall x \in X : f(x) = \lim_{n\to\infty} f_n(x) .$$

While *uniform convergence* can be understood by studying the *supremum metric* from Example A.2 e), the *pointwise convergence* can not be described by a metric! (We will see this fact later in the lecture!)

Weak Topologies: Similarly, there are situations in the study of infinite dimensional vector spaces in which one would like to study certain "weak" convergence properties (loosely comparable to pointwise convergence) which are not describable by metrics.

Uncountable Products: A very related situation: Sometimes one can realize a certain space as a subspace of an *uncountable* product. Those products however can not be described by a metric.

Compactifications: This is too abstract to be a motivating example, however there are situation in which one wants to make a space, say the natural numbers $\mathbb{N}$, compact. We could do this by embedding the natural numbers as $\{1/n \mid n \in \mathbb{N}\} \subseteq [0,1]$ and taking the closure. Note however, that not every continuous function $f : \mathbb{N} \to \mathbb{R}$ can be extended to a continuous function on this compactification. (Exercise: find an example!) If you want to have a compactification to which you can extend all continuous functions one has to give up a lot of other structural properties (you can not get such nice things for free). Most importantly those compactifications are not described by metrics.

Constructions in many algebraic areas: Sometimes it is useful to assign a certain *space* to an algebraic object (see ALGEBRAIC GEOMETRY and in particular the ZARISKI TOPOLOGY[1]). Those spaces are often not described by a metric.

Saving Headspace: In many situations one does not need to think about the special properties of metrics (even if they describe the structure of the space). We have seen that continuity, convergence and compactness are all completely determined by open sets alone. Many other properties also depend only on *topological information*, i. e. open sets. Sometimes it makes your work easier, the theory tidier and your proofs cleaner if you abstract away from unnessecary details.

Independently on whether you find the above remarks motivating or not, the history of mathematics has shown that topology is a very fruitful field with applications both in other fields of mathematics as well as other sciences.
A fitting intuition for topological reasoning is to do geometry but to ignore lengths and angles. This is perfectly exemplified by EULER'S solution of the Problem of the "Seven Bridges

[1] https://en.wikipedia.org/wiki/Zariski_topology

of Königsberg" (https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg) which could be seen as groundwork for both topology and graph theory.
So by copying this characterization as a definition we unders continuity, convergence and compactness in this topological space.

Chapter 1

Basic Notions

1.1 Viewpoints to Study Topological Spaces*

There are various different (equivalent) ways how one can capture the structure of a topological space. While the definition via open sets is nowadays the standard one (and the one we will be mostly working with), it is very illuminating to see from different viewpoints how the structure of a topological space emerges.
NOTE: it can be very confusing to study many equivalent definitions of a concept one has only heard the definition of. So do not pressure yourself to understand this section on the first read. In fact you may skip it until you feel like you have gotten a grasp on the standard definition of topological spaces. You can skip this chapter safely as any important definition made here will be repeated in the main part of the lecture notes. This chapter should serve as something to come back to over and over again. Especially with each new concept one can come back and ponder whether this concept could be also formulated from the other viewpoints. We will do this regularly.

1.1.1 The Standard Definition

Definition 1.1 (Topology)**.** A system of subsets $\mathcal{T} \subseteq \mathfrak{P}(X)$ is called a **topology (on X)** if

O1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

O2) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$. (*Closure under finite intersection*)

O3) For any subset $\mathcal{O} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{O} \in \mathcal{T}$. (*Closure under arbitrary union*)

The pair $(X, \mathcal{T})$ is called a **topological space**.

The key intuition is to think of the elements of a topology $\mathcal{T}$ as *open*. So the step from metric spaces to topological spaces is done by forgetting about the reason why certain sets are open. (Exercise: Prove that the system of open sets of a metric space is indeed a topology! (Cf. Proposition A.11)) We get a mapping from the class of metric spaces to the class of topological spaces

$$(X, d) \longmapsto (X, \{O \in \mathfrak{P}(X) \mid O \text{ is open with respect to the metric } d\})$$

assigning each metric space its *induced* topological space.
Using Lemma A.18 and Corollary A.22.1 we can define notions of convergence and continuity for topological spaces (in a way that the convergence and continuity in a metric space and its induced metric space coincide).

1.1.2 Alternative Viewpoints

Let $\mathcal{T}$ be a topology on X.

1.1.2.1 Closed Sets

Definition 1.2 (Closed Sets)**.** We say that A is closed in $(X, \mathcal{T})$ if $A^c \in \mathcal{T}$, i.e. if A^c is open.

By taking the complements we can go over to a system of closed sets, i.e.

$$\mathcal{T}_{\mathrm{cl}} := \{U^c \mid U \in \mathcal{T}\}$$

Lemma 1.3 (Structure of Closed Sets)**.** *$\mathcal{T}_{\mathrm{cl}}$ satisfies*

C1) $\emptyset \in \mathcal{T}_{\mathrm{cl}}$ and $X \in \mathcal{T}_{\mathrm{cl}}$.

C2) If $A, F \in \mathcal{T}_{\mathrm{cl}}$, then $A \cup F \in \mathcal{T}_{\mathrm{cl}}$.

C3) For any subset $K \subseteq \mathcal{T}_{\mathrm{cl}}$ we have $\bigcap K \in \mathcal{T}_{\mathrm{cl}}$.

Proof. Follows from De Morgan (Exercise!). □

Note that if $\mathcal{T}'_{\mathrm{cl}} \subseteq \mathfrak{P}(X)$ satisfies the properties stated in Lemma 1.3, then $\mathcal{T}' := \{F^c \mid F \in \mathcal{T}'_{\mathrm{cl}}\}$ is a topology. Therefore, we conclude:

Proposition 1.4 (Topological Spaces via Closed Sets)**.** *We can* equivalently *define topological spaces via systems of closed sets satisfying Lemma 1.3.* ■

(Exercise: What exactly does "equivalently" mean in the above Proposition 1.4?)

1.1.2.2 Kuratowski Closure Axioms

Recall the definition and basic properties of the closure in metric spaces laid out in Definition A.14 and Lemma A.16. For general topological spaces $(X, \mathcal{T})$ we can define a **closure operator** $\mathrm{cl}_{\mathcal{T}}$ associated to the topology $\mathcal{T}$.

Definition 1.5. For $A \subseteq X$ we define

$$\mathrm{cl}_{\mathcal{T}}(A) := \bigcap \{F \in \mathfrak{P}(X) \mid F^c \in \mathcal{T} \text{ (i. e. } F \text{ is closed) and } F \supseteq A\} \ .$$

Often we omit the subscript $\mathcal{T}$ if the choice of topology is clear. We then call $\mathrm{cl}(A)$ the **closure** of A. Another common notation for the closure of A is $\overline{A}$.

Trying to capture what a closure operator intuitively should satisfy we may define

Definition 1.6 (Kuratowski Closure Operator)**.** A map $c : \mathfrak{P}(X) \to \mathfrak{P}(X)$ is called a **Kuratowski closure operator** if

K1) $c(\emptyset) = \emptyset$.

K2) $A \subseteq c(A)$ for all $A \in \mathfrak{P}(X)$. (*Extensivity*)

K3) $c(c(A)) = c(A)$ for all $A \in \mathfrak{P}(X)$. (*Idempotence*)

K4) $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \in \mathfrak{P}(X)$. (*Compatibility with unions*)

Now if you consider a set to be closed if and only if it is its closure on can assign a system of closed sets to a closure operator. This gives us the following alternative characterization of topological spaces:

Proposition 1.7 (Topological Spaces via Kuratowski Closure Operators)**.** *$\mathrm{cl}_{\mathcal{T}}$ is a Kuratowski closure operator for any topology $\mathcal{T}$ on X. Conversely, for any Kuratowski closure operator c the family*

$$\mathcal{T}_c := \{A \in \mathfrak{P}(X) \mid c(A^c) = A^c\}$$

is a topology on X and the assignment

$$\mathcal{T} \longmapsto \mathrm{cl}_{\mathcal{T}} \quad \textit{and} \quad c \longmapsto \mathcal{T}_c$$

are mutually inverse.

Proof. **Topologies induce Kuratowski Closure Operators:** Let $\mathcal{T}$ be a topology on X.

K1) Then clearly $\emptyset^c = X \in \mathcal{T}$ so $\emptyset$ is closed. Therefore,

$$\mathrm{cl}_{\mathcal{T}}(\emptyset) = \bigcap \underbrace{\{F \in \mathfrak{P}(X) \mid F^c \in \mathcal{T} \text{ and } F \supseteq \emptyset\}}_{\ni \emptyset} = \emptyset\,.$$

K2) Now let $A \subseteq X$ be arbitrary. Then for any

$$C \in \{F \in \mathfrak{P}(X) \mid F^c \in \mathcal{T} \text{ and } F \supseteq A\}$$

we have *by definition* that $A \subseteq C$. Thus

$$A \subseteq \bigcap \{F \in \mathfrak{P}(X) \mid F^c \in \mathcal{T} \text{ and } F \supseteq A\} = \mathrm{cl}_{\mathcal{T}}(A)\,.$$

This proves the extensivity.

K3) As arbitray unions of open sets are open, De Morgan implies that arbitrary intersections of closed sets are closed, so $\mathrm{cl}_{\mathcal{T}}(A)$ is closed itself. (Thats why the definiton of "closure" is useful in the first places) So $\mathrm{cl}_{\mathcal{T}}(A)$ is a closed superset of A and thus

$$\mathrm{cl}_{\mathcal{T}}\left(\mathrm{cl}_{\mathcal{T}}(A)\right) = \bigcap \underbrace{\{F \in \mathfrak{P}(X) \mid F^c \in \mathcal{T} \text{ and } F \supseteq \mathrm{cl}_{\mathcal{T}}(A)\}}_{\ni \mathrm{cl}_{\mathcal{T}}(A)} \subseteq \mathrm{cl}_{\mathcal{T}}(A)\,.$$

Together with the extensivity this implies the idempotence.

K4) Now let $A, B \in \mathfrak{P}(X)$ be arbitrary. Observe that $\mathrm{cl}_{\mathcal{T}}$ is clearly monotone, e. g.

$$\mathrm{cl}_{\mathcal{T}}(A \cup B) \supseteq \mathrm{cl}_{\mathcal{T}}(A) \quad \text{and} \quad \mathrm{cl}_{\mathcal{T}}(A \cup B) \subseteq \mathrm{cl}_{\mathcal{T}}(B)\,.$$

So we have $\mathrm{cl}_{\mathcal{T}}(A \cup B) \supseteq \mathrm{cl}_{\mathcal{T}}(A) \cup \mathrm{cl}_{\mathcal{T}}(B)$. Conversely, observe that $\mathrm{cl}_{\mathcal{T}}(A) \cup \mathrm{cl}_{\mathcal{T}}(B)$ is a closed superset of $A \cup B$. Thus, $\mathrm{cl}_{\mathcal{T}}(A \cup B) \subseteq \mathrm{cl}_{\mathcal{T}}(A) \cup \mathrm{cl}_{\mathcal{T}}(B)$.

Kuratowski Closure Operators induce Topologies: Now let c be a Kuratowski closure operator. We show that

$$K_c := \{F \in \mathfrak{P}(X) \mid F = c(F)\}$$

satisfies Lemma 1.3. This immediatly implies, by Proposition 1.4 that

$$\mathcal{T}_c := \{A \in \mathfrak{P}(X) \mid A^c = c(A^c)\}$$

is a topology on X.

C1) By K1) we see that $\emptyset \in K_c$. By K2) we further see that $X \in K_c$.

C2) Now let $A, F \in K_c$. Then

$$A \cup F = c(A) \cup c(F) = c(A \cup F)\,.$$

So $A \cup F \in K_c$. This immediatly also implies monotonicity, i. e. $c(A) \subseteq c(B)$ whenever $A \subseteq B$.

C3) Now let $K \subseteq K_c$ be arbitrary. Note that $\bigcap K \subseteq F$ for any $F \in K$. Thus $c(\bigcap K) \subseteq c(F)$ for any $F \in K$. Therefore,

$$c\left(\bigcap K\right) \subseteq \bigcap_{F \in K} c(F) = \bigcap_{F \in K} F = \bigcap K$$

By extensivity of c we have $c\left(\bigcap K\right) = \bigcap K$ and thus $\bigcap K \in K_c$.

The mutual inverseness of the operators is an exercise! □

So we could equivalently define topological spaces via Kuratowski closure operators.

1.1.2.3 Hausdorff Neighbourhood Axioms

Again let $(X, \mathcal{T})$ be a topological space. Recall the definition of a neighbourhood in metric spaces (Definition A.13). We define

Definition 1.8 (Neighbourhood Filter)**.** Let $x \in X$ and $A \subseteq X$. We call A a **neighbourhood** of x if there is an set $U \in \mathcal{T}$, i. e. an open set, such that $x \in U \subseteq A$.
By $\mathfrak{U}_{\mathcal{T}}(x)$ we denote the system of all neighbourhoods of x and call $\mathfrak{U}_{\mathcal{T}}(x)$ called the **neighbourhood filter** of x.

Trying to capture abstractly what a (system of) neighbourhood should satisfy we might come up with

Definition 1.9 (Hausdorff Neighbourhood Filter Axioms)**.** A map

$$\mathfrak{U} : X \longrightarrow \mathfrak{P}\left(\mathfrak{P}\left(X\right)\right), \ x \longmapsto \mathfrak{U}(x)$$

is called a **Hausdorff neighbourhood filter operator** if

H1) $x \in U$ for any $U \in \mathfrak{U}(x)$ (*Neighbourhoods contain the point of which they are neighbourhoods*) and $X \in \mathfrak{U}(x)$ (*The whole space is always a neighbourhood*).

H2) For any two $U, V \in \mathfrak{U}(x)$ we have $U \cap V \in \mathfrak{U}(x)$. (*The intersection of two neighbourhoods is again a neighbourhood*)

H3) If $V \supseteq U$ for some $U \in \mathfrak{U}(x)$, then $V \in \mathfrak{U}(x)$. (*Supersets of neighbourhoods are neighbourhoods themselves*)

H4) For any $U \in \mathfrak{U}(x)$ there is $V \in \mathfrak{U}(x)$ such that $V \subseteq U$ and for any $y \in U$ we have $U \in \mathfrak{U}(y)$. (*Any neighbourhood contains an open neighbourhood*)

It turns out that for any topology the system of neighbourhood filters is a Hausdorff neighbourhood filter operator. Conversely, every Hausdorff neighbourhood filter operator induces a topology.

Proposition 1.10 (Topological Spaces via Hausdorff Neighbourhood Filters)**.** $\mathfrak{U}_{\mathcal{T}}$ *is a Hausdorff neighbourhood filter operator for any topology* $\mathcal{T}$ *on* X*. Conversely, for any Hausdorff neighbourhood filter operator* $\mathfrak{U}$ *the family*

$$\mathcal{T}_{\mathfrak{U}} := \{U \in \mathfrak{P}\left(X\right) \mid \forall x \in U : U \in \mathfrak{U}(x)\}$$

is a topology on X *and the assignment*

$$\mathcal{T} \longmapsto \mathfrak{U}_{\mathcal{T}} \quad \text{and} \quad \mathfrak{U} \longmapsto \mathcal{T}_{\mathfrak{U}}$$

are mutually inverse.

Proof. **Topologies induce Hausdorff Neighbourhood Filter Operators:** Let $\mathcal{T}$ be a topology. We show that $\mathfrak{U}_{\mathcal{T}} : X \to \mathfrak{P}\left(\mathfrak{P}\left(X\right)\right)$ is a Hausdorff neighbourhood filter operator. Let $x \in X$ be arbitrary.

H1) So let $U \in \mathfrak{U}_{\mathcal{T}}(x)$. Then there is an open $O \subseteq U$ with $x \in 0$. So $x \in U$. As X is open itself it is a neigbhourhood of all its points and thus $X \in \mathfrak{U}_{\mathcal{T}}(x)$.

H2) Let $U, V \in \mathfrak{U}_{\mathcal{T}}(x)$. Then there are open $O_U \subseteq U$ and $O_V \subseteq V$ with $x \in O_U$ and $x \in O_V$. Then $O := O_U \cap O_V$ is open an $O \subseteq U \cap V$ with $x \in O$. So $U \cap V \in \mathfrak{U}_{\mathcal{T}}(x)$.

H3) Let $U \in \mathfrak{U}_{\mathcal{T}}(x)$ and $V \supseteq U$. There is an open $O \subseteq U$ with $x \in O$. Clearly $O \subseteq V$ and so $V \in \mathfrak{U}_{\mathcal{T}}(x)$.

H4) Let $U \in \mathfrak{U}_{\mathcal{T}}(x)$. Then there is an open $O \subseteq U$ with $x \in O$. Now let $y \in O$. Then, as O is open, there is an open subset of O, namely O itself, containing y. So $O \in \mathfrak{U}_{\mathcal{T}}(y)$.

Hausdorff Neighbourhood Filter Operators induce Topologies: Let $\mathfrak{U}$ be an Hausdorff neighbourhood filter operator. The idea is to call a set open if it is a "neighbourhood" of each of its elements. So we define

$$\mathcal{T}_{\mathfrak{U}} := \{U \in \mathfrak{P}(X) \mid \forall x \in U : U \in \mathfrak{U}(x)\}$$

and have to prove, that its a topology.

O1) As the empty set has no elements the condition for $\emptyset \in \mathcal{T}_{\mathfrak{U}}$ is *vacuoulsy* satisfied. Conversely as X is always a neighbourhood, by H1) we also have that $X \in \mathcal{T}_{\mathfrak{U}}$.

O2) Now, let $U, V \in \mathcal{F}_{\mathfrak{U}}$. For $x \in U \cap V$ we have that $x \in U$ and $x \in V$ and therefore $U \in \mathfrak{U}(x)$ and $V \in \mathfrak{U}(x)$. Thus $U \cap V \in \mathfrak{U}(x)$, by H2), and therefore $U \cap V \in \mathcal{F}_{\mathfrak{U}}$.

O3) Finally, let $\mathcal{O} \subseteq \mathcal{T}_{\mathfrak{U}}$ be a family of open sets. Let $U := \bigcup \mathcal{O}$. For $x \in U$ there is $O \in \mathcal{O}$ with $x \in O \in \mathcal{T}_{\mathfrak{U}}$. Then, by definition of $\mathcal{T}_{\mathfrak{U}}$, we have $O \in \mathfrak{U}(x)$. By H3) we have $\bigcup \mathcal{O} \in \mathfrak{U}(x)$. As $x \in U$ was arbititrary, we have $\bigcup \mathcal{C} \in \mathcal{T}_{\mathfrak{U}}$.

So $\mathcal{T}_{\mathfrak{U}}$ is a topology.

Now the careful reader might wonder, what is the use of H4)? We have not yet used it (except shown that it is satisfied by the neighbourhood filters induced by topologies). Systems of neighbourhoods **not** satisfying H4) still induce topologies. So is H4) already implied by the other axioms? Unlike wrongly claimed in a previous version of those lecture notes (and a source I checked) the naive proof of this does not hold up to scrutiny. While a system of "neighbourhoods" satisfying H1), H2) and H3) induces a topology which in turn induces a system of "neighbourhoods" that satisfies H1), H2), H3) **and** H4), those systems of "neighbourhoods" need not be the same. In fact they are the same if and only if the original one satisfies H4) itself.

So H4) is the crucial ingredient in the proof that those assignments

$$\mathcal{T} \longmapsto \mathfrak{U}_{\mathcal{T}} \quad \text{and} \quad \mathfrak{U} \longmapsto \mathcal{T}_{\mathfrak{U}}$$

are mutually inverse.

$\mathcal{T}_{\mathfrak{U}_{\mathcal{T}}} = \mathcal{T}$: Let $\mathcal{T}$ be a topology. Then $U \in \mathcal{T}_{\mathfrak{U}_{\mathcal{T}}}$ if and only if U is a neigbourhood of all of its points. This is equivalent to $U = \int(U)$ which is equivalent to $U \in \mathcal{T}$.

$\mathfrak{U}_{\mathcal{T}_{\mathfrak{U}}} = \mathfrak{U}$: Let $x \in X$. Let $U \in \mathfrak{U}(x)$. Then, by H4), there is $V \in \mathfrak{U}(x)$ with $V \subseteq U$ which is a neighbourhood of all its points. Thus $V \in \mathcal{T}_{\mathfrak{U}}$. As $x \in V \subseteq U$ we see that U is a neighbourhood of x in the topology $\mathcal{T}_{\mathfrak{U}}$. Thus $U \in \mathfrak{U}_{\mathcal{T}_{\mathfrak{U}}}(x)$.

Conversely, let $U \in \mathfrak{U}_{\mathcal{T}_{\mathfrak{U}}}(x)$. Then there is $V \subseteq U$ with $V \in \mathcal{T}_{\mathfrak{U}}$ and $x \in V$. As $V \in \mathcal{T}_{\mathfrak{U}}$ we have that $V \in \mathfrak{U}(y)$ for any $y \in V$. As $x \in V$, we have $V \in \mathfrak{U}(x)$. As $V \in \mathfrak{U}(x)$, H3) implies $U \in \mathfrak{U}(x)$.

A careful examination of the above proof shows that while not every system of "neighbourhoods" satisfiying H1), H2) and H3) satisfies H4) it can be **extended** into one (by going to $\mathfrak{U}_{\mathcal{T}_{\mathfrak{U}}}$. This is the case as $\mathfrak{U}(x) \subseteq \mathfrak{U}_{\mathcal{T}_{\mathfrak{U}}}(x)$ does not require H4). □

So we can think about topologies by thinking about neighbourhoods. This offers a good intuition.

1.1.2.4 Honorable Mentions

The following fall short of equivalently definining topologies (we will see over the course of the lecture why this is the case and how to fix this), however they still offer valuable insights into how one can understand topologies.

Again, let $(X, \mathcal{T})$ be a topological space. Recall Definition A.17 and Lemma A.18. This allows us to generalize convergence from metric spaces to arbitrary topological spaces.

Definition 1.11 (Convergence)**.** Let $(x_n)_{n\in\mathbb{N}}$ a sequence and $x \in X$. We say that x is a **limit** of $(x_n)_{n\in\mathbb{N}}$ or, equivalently, that $(x_n)_{n\in\mathbb{N}}$ **converges** to x if

$$\forall U \in \mathfrak{U}(x) : \exists N \in \mathbb{N} : \forall n > N : x_n \in U .$$

We will write $x_n \xrightarrow{n\to\infty} x$.

Remark 1.12. Note that I intentionally spoke of "a limit". We will soon see that in general topological spaces the limit of a sequence needs not be unique.

One might have the intuition, that convergence of sequences somehow determines the structure of the topology. This intuition is leading in the right direction. However, we will see, that convergence of *sequences* alone is not sufficient, as some topological spaces are too complicated. We will later develop the tools to fix this intuition.
Similarly, one could try to determine the topology on a space by considering continuous functions.

Definition 1.13 (Continuity)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and $x \in X$. A function $f : X \to Y$ is called **continuous in** x if

$$\forall V \in \mathfrak{U}_{\mathcal{T}_Y}(f(x)) : f^{-1}(V) \in \mathfrak{U}_{\mathcal{T}_X}(x) ,$$

i. e. if preimages of neighbourhoods are neighbourhoods.
f is called **continuous** if f is continuous in x for all $x \in X$.

However, simply considering the set of continuous functions $\{f : X \to \mathbb{R} \mid f \text{ is continuous }\}$ does not determine the topology on X in general. The reason here is again, that the topological space might be complicated in a manner that can not be captured in $\mathbb{R}$.
A similarly not completely working intuition is to consider the *continuous* functions.

1.2 Definition and First Examples

Definition 1.14 (Topology)**.** A system of subsets $\mathcal{T} \subseteq \mathfrak{P}(X)$ is called a **topology (on X)** if

O1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

O2) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$. (*Closure under finite intersection*)

O3) For any subset $\mathcal{O} \subseteq \mathcal{T}$ we have $\bigcup \mathcal{O} \in \mathcal{T}$. (*Closure under arbitrary union*)

The pair $(X, \mathcal{T})$ is called a **topological space**.

Definition 1.15 (Open and Closed Sets)**.** By *definition* we call all sets $U \in \mathcal{T}$ **open**. If a $F^c \in \mathcal{T}$, then we call F **closed**.

Remark 1.16. As with metric spaces: There are sets which are neither open nor closed. Sets which are only one but not the other. And sets which are both open and closed.

Example 1.17. Let X be a set.

i) Every metric *induces* a topology. Let d be a metric on X, then

$$\mathcal{T}_d := \{O \in \mathfrak{P}(X) \mid O \text{ is open with respect to } d\} .$$

With all definitions we will make it is an useful exercise to check that, if the notion is also defined for metric spaces both definitions coincide whether applied to the metric or to the induced topology!

ii) The whole power set $\mathfrak{P}(X)$ is a topology. This is an important example and gets the name **discrete topology**. (Exercise: By which metric is this topology induced?)

iii) On the other end of the spectrum we have the **indiscrete** or **trivial topology** given by $\{\emptyset, X\}$. It is not induced by a metric if X has at least two elements. (Exercise: Why?)

iv) The **co-finite** topology is given by

$$\mathcal{T}_{\text{cf}} := \{A \in \mathfrak{P}(X) \mid A^c \text{ is finite }\} \cup \{\emptyset\} .$$

Exercise: Show that this is a topology. Exercise: Is it induced by a metric?

v) The **co-countable** topology is given by

$$\mathcal{T}_{\text{cc}} := \{A \in \mathfrak{P}(X) \mid A^c \text{ is countable }\} \cup \{\emptyset\} .$$

Exercise: Show that this is a topology. Exercise: Is it induced by a metric?

We can now extend the definitions we know from metric spaces to all topological spaces.

Definition 1.18. Let $(X, \mathcal{T})$ be a topological space Let $A \subseteq X$ and $a \in A$.

- We call A **closed** if $A^c \in \mathcal{T}$, i. e. if A^c is open.
- We call A a **neighbourhood** of a if there is $O \in \mathcal{T}$ with $a \in O$ and $O \subseteq A$.
- The **interior** of A is given by

$$\operatorname{int}(A) := \{x \in X \mid A \text{ is neighbourhood of} x\} = \bigcup \{O \in \mathcal{T} \mid O \subseteq A\} .$$

- The **closure** of A is given by

$$\begin{aligned}\operatorname{cl}(A) :=& \{x \in X \mid A^c \text{ is not a neighbourhood of} x\} \\ =& \bigcap \{F \in \mathfrak{P}(X) \mid F^c \in \mathcal{T} \text{ and } A \subseteq F\}\end{aligned}$$

- The **boundary** of A is given by

$$\partial A := \operatorname{cl}(A) \setminus \operatorname{int}(A) .$$

- A is called **dense** if $\operatorname{cl}(A) = X$ and **nowhere dense** if $\operatorname{int}(\operatorname{cl}(A)) = \emptyset$.

(Exercise: Prove the equalities claimed in the definition of the closure and interior and state and prove an analogue of Lemma A.16.)

1.3 Convergence

Let $(X, \mathcal{T})$ be a topological space.

Definition 1.19 (Convergence)**.** Let $A \subseteq X$ and $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence and $x \in X$.

- We call x a **accumulation point** of the subset A if $V \cap A \setminus \{x\} \neq \emptyset$ for any $V \in \mathfrak{U}(x)$.
- We call x an **cluster point** of the sequence $(x_n)_{n\in\mathbb{N}}$ if for any $V \in \mathfrak{U}(x)$ and any $N \in \mathbb{N}$ there is $n > N$ such that $x_n \in V$.
- We say that the sequence $(x_n)_{n\in\mathbb{N}}$ **converges** to x if for any $V \in \mathfrak{U}(x)$ there is $N \in \mathbb{N}$ such that for all $n > N$ we have $x_n \in V$.

Exercise:

a) Find a sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x but x is no accumulation point of $\{x_n \mid n \in \mathbb{N}\}$.

b) Find a sequence $(x_n)_{n\in\mathbb{N}}$ such that x is the only cluster point of $(x_n)_{n\in\mathbb{N}}$ but $(x_n)_{n\in\mathbb{N}}$ does not converge to x.

c) Show that
$$\mathrm{cl}(A) = A \cup \{x \in X \mid x \text{ is accumulation point of } A\} \ .$$

d) Find (a topological space $(X, \mathcal{T})$ and) a sequence $(x_n)_{n\in\mathbb{N}}$ and a point $x \in X$ such that x is an accumulation point of $\{x_n \mid n \in \mathbb{N}\}$ but no cluster point of $(x_n)_{n\in\mathbb{N}}$.

e) Describe the convergence with respect to the discrete topology.

f) Describe the convergence with respect to the indiscrete topology.

g) Describe the convergence with respect to the co-finite topology.

h) Describe the convergence with respect to the co-countable topology.

Appendix A

Metric Spaces

For the union of sets there are two closely related notions. If $(A_i)_{i\in I}$ is a family of sets over some index set I, then

$$\bigcup_{i\in I} A_i = \{x \mid \exists i \in I : x \in A_i\} \ .$$

Similarly if $\mathcal{A}$ is a set of sets then

$$\bigcup \mathcal{A} = \{x \mid \exists A \in \mathcal{A} : x \in A\} \ .$$

Analogous notation will be used for intersections.

A.1 Definition and Examples

Let X be any set.

Definition A.1 (Metric Space)**.** A **metric** (or **distance function**) on X is a map

$$d : X \times X \longrightarrow [0, \infty),\ (x, y) \longmapsto d(x, y)$$

which is **positive definite**, i. e.

$$d(x, y) = 0 \Longleftrightarrow x = y\,,$$

symmetric, i. e.

$$d(x, y) = d(y, x)$$

and satisfies the **triangle inequality**, i. e.

$$d(x, z) \leqslant d(x, y) + d(y, z)$$

for all $x, y, z \in X$. If d is a metric on X one calls the pair (X, d) a **metric space**.

One very important notation (which varies between textbooks) is that of an "ε-ball". For any $x \in X$ we define

$$B_{\varepsilon,d}(x) := \{y \in X \mid d(x, y) < \varepsilon\}$$

and call it the **open ε-ball around** x. If the choice of metric is clear we will simply write $B_\varepsilon(x)$.

Example A.2. a) The arguably most intuitive example is that of the real number line equipped with $d(x, y) = |x - y|$. Then $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

b) Similarly, the usual norm on $\mathbb{R}^n$ yields a metric on those spaces. To be precise, we define

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}\,.$$

While the positive definiteness and the symmetry of d are straightforward, the proof of the triangle inequality requires some work. It follows from the CAUCHY-SCHWARZ INEQUALITY, which states that

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i \leqslant \sqrt{\sum_{i=1}^{n} v_i^2} \cdot \sqrt{\sum_{i=1}^{n} w_i^2} = \|v\| \cdot \|w\| \ . \tag{A.1}$$

One can calculate that

$$\begin{aligned}
d(x,z)^2 &= \sum_{i=1}^{n}(z_i - x_i)^2 = \sum_{i=1}^{n}((z_i - y_i) + (y_i - x_i))^2 \\
&= \sum_{i=1}^{n}(z_i - y_i)^2 + 2\sum_{i=1}^{n}(z_i - y_i)(y_i - x_i) + \sum_{i=1}^{n}(y_i - x_i)^2 \\
&\overset{(A.1)}{\leq} \sum_{i=1}^{n}(z_i - y_i)^2 + 2\sqrt{\sum_{i=1}^{n}(z_i y_i)^2} \cdot \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} + \sum_{i=1}^{n}(y_i - x_i)^2 \\
&= \left(\sqrt{\sum_{i=1}^{n}(z_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}\right)^2 = \left(d(z,y) + d(y,x)\right)^2 \ .
\end{aligned}$$

The "open ball" $B_\varepsilon(x)$ is now exactly an open ball of radius ε around x.

c) The above example immediatly generalizes to all normed vector spaces. Assume that $\|\cdot\|$ is a norm on a vector space V. Then

$$d : V \times V \longrightarrow [0,\infty),\ (v,w) \longmapsto \|v - w\|$$

is a metric on V. Here the properties of the metric immediatly follow from the properties of a norm (in fact, the homogeneity is not needed). We call those metrics **norm-induced**.

d) On $\mathbb{R}^n$ one can define a multitude of metrics that are all "similar" in some sense (what are the mathematical consequences of this similarity?). For example, both

$$d_{\text{sup}} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty),\ (v,w) \longmapsto \max_{i=1}^{n} |v_i - w_i|$$

and

$$d_1 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,\infty),\ (v,w) \longmapsto \sum_{i=1}^{n} |v_i - w_i|$$

are metrics on $\mathbb{R}^n$ (prove this as an exercise!).

Interestingly, the "open balls" look differently under those metrics. In fact, it holds that

$$B_{1,d_{\text{sup}}}(0) = (0,1)^n$$

and B_{1,d_1} is the rotatated and shrunken hypercube $\{\pm e_k \mid k \in \{1,\dots,n\}\}$ for the standard basis $\{e_1,\dots,e_n\}$ with $e_{kn} = \delta_{k,n}$. (Exercise: It is a nice exercise to draw those "open balls" and understand their structure!)

e) Let X be any set and (Y, d_Y) a metric space. The most important case will be $Y = \mathbb{R}$ equipped with the euclidean metric. Consider the **bounded** functions

$$\mathcal{F}_b(X,Y) := \left\{ f : X \to Y \,\middle|\, \sup_{x \in X} d_Y(f(x), f(x_0)) < \infty \right\}$$

and observe that the above definition does not depend on the choice of $x_0 \in X$ (Exercise: prove this from the triangle inequality!). We then use the metric d_Y on Y in order to define the **supremum metric** $d_{\sup}$ on $\mathcal{F}_b(X,Y)$, given by

$$d_{\sup} : \mathcal{F}_b(X,Y) \times \mathcal{F}_b(X,Y) \to [0,\infty),\ (f,g) \longmapsto \sup_{x \in X} d_Y(f(x), g(x)) .$$

(Exercise: It is a good exercise to check that $d_{\sup}$ is indeed a metric and to understand whats the role of the restriction to bounded functions!).

f) The arguably *simplest* metric that one can define on any set X is the **discrete metric**. We simply define

$$d : X \times X \longrightarrow [0,\infty),\ (x,y) \longmapsto \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} .$$

This metric simply says that any point has distance one to any other point.

The *triangle inequality* can be equivalently written as the **reverse triangle inequality**.

Lemma A.3. *A symmetric map $d : X \times X \to [0,\infty)$ satisfies the triangle inequality if and only if*

$$|d(z,x) - d(x,y)| \leqslant d(z,y)$$

for any $x, y, z \in X$.

Proof. Let $x, y, z \in X$. The reverse riangle inequality immediatly implies

$$d(z,x) - d(x,y) \leqslant d(z,y) .$$

Adding $d(x,y)$ on both sides and using symmetry yields the triangle inequality.
Conversely, we can conclude from the triangle inequality that $d(x,y) \leqslant d(x,z) + d(y,z)$ and therefore $-d(x,z) + d(x,y) \leqslant d(x,y)$. Similarly, $d(x,z) \leqslant d(x,y) + d(y,z)$ and thus $d(x,z) - d(y,z) \leqslant d(x,y)$. Together this yields the reverse triangle inequality. ☐

A.2 Constructions of Metrics

With the supremum metric on the space of bounded functions we have already seen an example of how one can take a metric on one set and have it induce a metric on the other set. We will now see several more such constructions.

Definition A.4 (Subspace of a Metric Space)**.** Let $A \subseteq X$ be a subset of a metric space (X,d). Obviously we can define

$$d|_{A \times A} : A \times A \longrightarrow [0,\infty),\ (a,a') \longmapsto d(a,a')$$

by simply restricting. We will call $(A, d|_{A \times A})$ a **subspace** of (X,d) and often simply write (A,d).

Remark A.5 (Finite Product of Metric Spaces)**.** Let $(X_1,d_1), \ldots, (X_n,d_n)$ be a finite family of metric spaces. Then we can define a metric d on the product set $\prod_{i=1}^n X_i$ by

$$d : \prod_{i=1}^n X_i \times \prod_{i=1}^n X_i \longrightarrow [0,\infty),\ ((x_i)_{i=1}^n, (y_i)_{i=1}^n) \longmapsto \sum_{i=1}^n d_i(x_i,y_i) .$$

Similary, we can define

$$d' : \prod_{i=1}^{n} X_i \times \prod_{i=1}^{n} X_i \longrightarrow [0,\infty),\ ((x_i)_{i=1}^n, (y_i)_{i=1}^n) \longmapsto \sqrt{\sum_{i=1}^{n} (d_i(x_i, y_i))^2}\,.$$

(Exercise: Note that there are several methods of defining a "product metric". Contemplate the reasons for this and, if you have already seen the construction of the product topology, how topology avoids this non-uniqueness!)

Remark A.6 (Countably Infinite Product of Metric Spaces)**.** Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a countably infinite family of metric spaces where $\sup_{x,x'\in X_n} d_n(x,x') < 1$ for any $n \in \mathbb{N}$. Then we can define a metric d on the product set $\prod_{n\in\mathbb{N}} X_n$ by

$$d_{\text{ptw}} : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n \longrightarrow [0,\infty),\ ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \longmapsto \lim_{N\to\infty} \sum_{n=1}^{N} 2^{-n} d_n(x_n, y_n)\,.$$

Note that while the cartesian product of sets treats every factor equally, this definition of a product metric introduces an inhomogenity / assymmetry in the different coordinates. (Exercise: This metric induces *pointwise convergence* (on a countable set)). (Exercise: Convince yourself that this is necessary and, once you know about product topology, contemplate how topology circumvents this issue!)
Another metric on the countably infinite product is (assuming certain boundedness assumptions (Exercise!)) given by

$$d_{\sup} : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n \longrightarrow [0,\infty),\ ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \longmapsto \sup_{n\in\mathbb{N}} d_n(x_n, y_n)\,.$$

This metric is a generalization of the supremum metric (Example A.2 *(e)*). This metric is *qualitatively* distinct from the metric d_{ptw}.

Definition A.7 (Pullback of Metrics)**.** Let $f : X \to Y$ be any map and d_Y a metric on Y. We can use f to define a symmetric $d_X := f^* d_Y$ on X which satisfies the triangle inequality by

$$d_X : X \times X \longrightarrow [0,\infty),\ (x, x') \longmapsto d_Y(f(x), f(x'))\,.$$

If f is injective then d_X is a metric.

Remark A.8. With more work and certain assumptions it is also possible to define a metric on a quotient space.

A.3 Open and Closed Subsets of Metric Spaces

Let (X, d) be a metric space.

Definition A.9 (Open and Closed Sets in Metric Spaces)**.** A subset $U \subseteq X$ is called **open** if for any $x \in U$ there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. A subset $F \subseteq X$ is called **closed** if F^c is open.

Remark A.10. Note that in the context of topology "open" and "closed" are not contradictory terms. A set can very well be both open and closed. The trivial example is the empty set which is open and closed in any metric space. Less trivial is any subset of a set equipped with the discrete metric. (Exercise: Prove that any subset of a set equipped with the discrete metric is open and closed!)
Similarly, there are subsets that are *neither* open nor closed. This is for example true for the set $[0, 1)$ in the metric space $\mathbb{R}$ with the euclidean metric.

Proposition A.11. *The empty set $\emptyset$ and the whole space X are open. Finite intersections of open sets are open. Arbitrary unions of open sets are open.*

Proof. This is a very good exercise! □

Lemma A.12. *The "open ball" $B_\varepsilon(x)$ is indeed open for any $x \in X$ and $\varepsilon > 0$.*

Proof. This is a consequence of the triangle inequality.
Let $x \in X$ and $\varepsilon > 0$. We have to show that for any $y \in B_\varepsilon(x)$ there is a $\delta > 0$ such that $B_\delta(y) \subseteq B_\varepsilon(x)$. Let $a := d(x, y) < \varepsilon$. Then $\delta := \varepsilon - a > 0$ and for $z \in B_\delta(y)$ we have

$$d(z, x) \leqslant d(z, y) + d(y, z) < \delta + a = \varepsilon \,.$$ □

The following is basically the theory of the *topology of metric spaces*. We will study topology much more generally, however this setting offers great intuition for the more general situation.

Definition A.13 (Neighbourhoods in Metric Spaces)**.** Let $A \subseteq X$ and $x \in A$. We say that A is a **neighbourhood** of x if there is an open set $U \subseteq A$ such that $x \in U$.
Let $x \in X$ then by

$$\mathfrak{U}(x) := \{A \subseteq X \mid A \text{ is neighbourhood of } x\}$$

define the **neighbourhood filter** of x. (The usage of the word filter will be explained later in the lecture.)

The neighbourhood filter is the collection of all neighbourhoods of a point. Conversely, we can also consider the set of all points for which a given set is a neighbourhood.

Definition A.14 (Interior, Closure and Boundary in Metric Spaces)**.** The **interior** of a subset $A \subseteq X$ is given by

$$\operatorname{int}(A) := \{x \in X \mid A \text{ is neighbourhood of } x\} = \{x \in X \mid A \in \mathfrak{U}(x)\} \;.$$

Another common notation is $\operatorname{int}(A) = A^o$.
The **closure** of A is given by

$$\operatorname{cl}(A) := \{x \in X \mid A^c \text{ is no neighbourhood of } x\} = \{x \in X \mid A^c \notin \mathfrak{U}(x)\} \;.$$

(Exercise: Prove directly that $\operatorname{cl}(A) = \operatorname{int}(A^c)^c$!) Another common notation is $\operatorname{cl}(A) = \overline{A}$.
The **boundary** of A is defined as $\partial A := \operatorname{cl}(A) \setminus \operatorname{int}(A)$. (Open Exercise: Is the name fitting?)
The subset A is called **dense** if $\operatorname{cl}(A) = X$. Furthermore, A is called **nowhere dense** if $\operatorname{int}(\operatorname{cl}(A)) = \emptyset$.

Example A.15. a) In $\mathbb{R}$ with the euclidean metric the notions of open and closed coincide with the one we know from Analysis 1.

b) Let $A \subseteq \mathbb{R}^2$ be defined as

$$A := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; 0 \leqslant x < 1, 0 \leqslant y < 1\right\} = [0, 1) \times (0, 1] \,.$$

Then

$$\begin{aligned} \operatorname{int}(A) &= (0, 1)^2 \\ \operatorname{cl}(A) &= [0, 1]^2 \\ \partial A &= (\{0, 1\} \times [0, 1]) \cup ([0, 1] \times \{0, 1\}) \end{aligned}$$

(Exercise: Check and draw this!)

c) The rational numbers are dense in $\mathbb{R}$ with the euclidean metric.

d) Finite subsets are nowhere dense in $\mathbb{R}$ with the euclidean metric.

Lemma A.16. *Let $A \subseteq X$. Then ...*

a) ... the interior of A is open.

b) ... the closure of A is closed.

c) ...

$$\operatorname{int}(A) = \bigcup \{U \subseteq A \mid U \text{ is open } \} \,.$$

d) ...

$$\operatorname{cl}(A) = \bigcap \{F \supseteq A \mid F \text{ is closed } \} \,.$$

e) ... A is open if and only if $\operatorname{int}(A) = A$.

f) ... A is closed if and only if $\operatorname{cl}(A) = A$.

g) ... $\operatorname{int}(A)^c = \operatorname{cl}(A^c)$.

h) ... $\operatorname{int}(A^c) = \operatorname{cl}(A)^c$.

i) ... $\partial A = \partial(A^c)$.

(Exercise: Use c) to make sense out of "The interior is the biggest open subset". Similarly, use d) to make sense out of "The closure is the smallest closed superset".

Proof. We only prove c) and d, as those imply the other results (Exercise!).
Let $B := \bigcup \{U \subseteq A \mid U \text{ is open } \}$. Then B is open and $B \subseteq A$. Therefore, any $x \in B$ has B as a neighbourhood which is contained in A. So $B \subseteq \operatorname{int}(A)$. Conversely, let $x \in \operatorname{int}(A)$. Then there is a neighbourhood V of x which is contained in A. Furthermore, there is an open set $O \subseteq V$ with $x \in O$. So $x \in O \subseteq \bigcup \{U \subseteq A \mid U \text{ is open } \} = B$.
Now let $C := \bigcap \{F \supseteq A \mid F \text{ is closed } \}$. Then C is closed and $C \supseteq A$. Therefore, any $x \in C^c$ has C^c as a neighbourhood which is contained in A^c and so $x \notin \operatorname{cl}(A)$. Thus, $C^c \subseteq \operatorname{cl}(A)^c$ which is equivalent to $\operatorname{cl}(A) \subseteq C$. Conversely, let $x \in \operatorname{cl}(A)^c$. Then A^c is a neigbourhood of x and therefore there is $U \subseteq A^c$ open with $x \in U$. Then $x \notin U^c$. As U^c is a closed superset of A this implies $x \notin C$. So $\operatorname{cl}(A)^c \subseteq C^c$ which is equivalent to $C \subseteq \operatorname{cl}(A)$.
For the remaining exercises I will now give hints:

a) and b) can be proven by the fatcs that unions of open sets are open and intersections of closed sets are closed.

e) and f) follows by a) and b) and by definition of closure and interior.

g) and h) is a consequence of the De Morgan laws.

i) follows by De Morgan and g) and h). □

A.4 Convergence

Again let (X, d) be a metric space.

Definition A.17 (Convergence in Metric Spaces)**.** Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence and $x \in X$. We say that $(x_n)_{n\in\mathbb{N}}$ **converges** to x (or that x is a **limit point** of $(x_n)_{n\in\mathbb{N}}$) if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : d(x_n, x) < \varepsilon \,.$$

In this case we write $x = \lim_{n\to\infty} x_n$ and $x_n \xrightarrow{n\to\infty} x$.

Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence and $x \in X$.
Convergence *only depends on open sets*! This is one of the major facts that the abstract theory of topology builds upon.

Lemma A.18. *Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence and $x \in X$. Then $x = \lim_{n\to\infty} x_n$ if and only if for any neigbourhood $U \in \mathfrak{U}(x)$ there is $N \in \mathbb{N}$ such that for all $n > N$ we have $x_n \in U$.*

Proof. This is a very instrumental exercise. □

Lemma A.19. *Every sequence has at most one limit point.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence and $x, y \in X$. Assume that x and y are both limit points of $(x_n)_{n\in\mathbb{N}}$ and $x \neq y$. Then $a := d(x, y) > 0$. But there is $N_x \in \mathbb{N}$ such that for all $n > N_x$ we have $d(x_n, x) < \frac{a}{2}$. Similarly, there is $N_y \in \mathbb{N}$ such that for all $n > N_y$ we have $d(x_n, y) < \frac{a}{2}$. Now for $n > \max(N_x, N_y)$ we have $d(x, y) \leqslant d(x, x_n) + d(x_n, y) < a = d(x, y)$. A contradiction. □

Lemma A.20. *For any $A \subseteq X$ we have*

$$\operatorname{cl}(A) = \left\{ x \in X \;\middle|\; \exists (x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}} : x = \lim_{n\to\infty} x_n \right\}$$

Proof. Let $x \in \operatorname{cl}(A)$ and let $n \in \mathbb{N}$. As A^c is no neighbourhood of x we must have that $B_{\frac{1}{n}}(x) \cap A \neq \emptyset$. So pick $x_n \in B_{\frac{1}{n}}(x) \cap A$. Then clearly $A \ni x_n \xrightarrow{n\to\infty} x$.
Conversely let $x = \lim_{n\to\infty} x_n$ for some $(x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. Let U be any neighbourhood of x then there is $N \in \mathbb{N}$ such that for $n > N$ we have $x_n \in U$. As $x_n \in A$ we have $A \cap U \neq \emptyset$. In particular $U \neq A^c$. So A^c is no neighbourhood of x and thus $x \in \operatorname{cl}(A)$. □

(Exercise: Conclude that A is closed if and only if it contains all limits of all its converging sequences!)

A.5 Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a map.

Definition A.21 (Continuity in Metric Spaces)**.** For $x \in X$ we say that f **is continuous in** x if

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x' \in X : d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon \,.$$

We call f **continuous** if f is continuous in all $x \in X$.
If f is bijective and *both* f and f^{-1} are continuous, then f is called a **homeomorphism**.

Proposition A.22 (Equivalent Characterizations of Continuity)**.** *Let $x \in X$. The following are equivalent:*

(i) *f is continuous in x.*

(ii) *For every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \,.$$

(iii) *Let U be a neighbourhood of $f(x)$, then $f^{-1}(U)$ is a neighbourhood of x.*

(iv) *For every sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ with $x = \lim_{n\to\infty} x_n$ we have*

$$f(x) = \lim_{n\to\infty} f(x_n) \,.$$

In this case we call f ***sequentially continuous****.*

Proof. We show

$$(i) \iff (ii) \implies (iii) \implies (iv) \implies (ii)\,.$$

(i) is equivalent to (ii): This is just a reformulation. (Exercise!)

(ii) implies (iii): Let $U \subseteq Y$ be a neighbourhood of $f(x)$. Then there is $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq U$. Now by (ii), there is $\delta > 0$ such that

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq U\,.$$

Thus $B_\delta(x) \subseteq f^{-1}(U)$ and so $f^{-1}(U)$ is a neighbourhood of x.

(iii) implies (iv): Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence with $x = \lim_{n\to\infty} x_n$. Let U be a neighbourhood of $f(x)$. By (iii), $f^{-1}(U)$ is a neighbourhood of x. Thus, there is $N \in \mathbb{N}$ such that for all $n > N$ we have $x_n \in f^{-1}(U)$. Thus $f(x_n) \in U$ for $n > N$. As U was arbitrary this shows that $f(x) = \lim_{n\to\infty} f(x_n)$.

(iv) implies (ii): We proceed by contraposition. Assume that there is $\varepsilon > 0$ such that for any $\delta > 0$ we have

$$f(B_\delta(x)) \not\subseteq B_\varepsilon(f(x))\,.$$

Then, for any $n \in \mathbb{N}$ there exists $x_n \in B_{\frac{1}{n}}(x)$ such that $f(x_n) \notin B_\varepsilon(x)$. Thus $x = \lim_{n\to\infty} x_n$ and $f(x) \neq \lim_{n\to\infty} f(x_n)$. So f is not sequentially continuous. □

Corollary A.22.1 (*preimages of open sets are open*)**.** *f is continuous if and only if for every open set $U \subseteq Y$ the preimage $f^{-1}(U)$ is open in X.*

Proof. Exercise! □

Again we see that the continuity of functions is completely determined by open sets! The idea of topology is to abstract away the notion of a metric and just consider *open sets* (which then are open *by definition*).

A.6 Compactness

We directly choose a definition that only considers *open sets*.

Definition A.23 (Compact Metric Spaces)**.** A metric space (X, d) is called **compact** if for any family of open sets $\mathcal{U}$ with $\bigcup \mathcal{U} = X$ there is a *finite* subfamily $\{U_1, \dots, U_n\} \subseteq \mathcal{U}$ such that $\bigcup_{i=1}^n U_i = X$. (*Open covers have finite subcovers.*)

Theorem A.24 (Cantor Intersection Theorem for Metric Spaces)**.** *Let (X, d) be compact and let $\mathcal{A}$ be a family of closed subsets such that for any finite subfamility $\mathcal{A}'$ we have $\bigcap \mathcal{A}' \neq \emptyset$ (*finite intersection property*). Then $\bigcap \mathcal{A} \neq \emptyset$. (*If finite intersections of a family of closed subsets are non-empty, then the intersection over the whole family is non-empty aswell!*)*

Proof. The proof is just a reversal of the definition of compactness. Assume that $\bigcap \mathcal{A} = \emptyset$. Let $\mathcal{U} := \{A^c \mid A \in \mathcal{A}\}$. Then $\mathcal{U}$ is a family of open sets and $\bigcup \mathcal{U} = (\bigcap \mathcal{A})^c = \emptyset^c = X$ by de Morgan. So $\mathcal{U}$ is an open cover. Thus there is a finite $\{U_1, \dots, U_n\} \subseteq \mathcal{U}$ with $\bigcup_{i=1}^n U_i = X$. Now $\mathcal{A} \ni A_i := U_i^c$ and $\bigcap A_i = (\bigcup U_i)^c = X^c = \emptyset$ by de Morgan. So there is a finite subfamily $\mathcal{A}' = \{A_1, \dots, A_n\}$ with $\bigcap \mathcal{A}' = \emptyset$. A contradiction. □

In metric spaces the following characterization holds:

Proposition A.25. *(X, d) is compact if and only if every sequence has a converging subsequence. (*We call such spaces **sequentially compact**.*)*

Proof. Let (X, d) be compact and $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ any sequence. Consider the *tail* $T_N := \{x_n \mid n > N\}$. Then $\mathrm{cl}(T_N)$ is closed and (Exercise!)

$$\bigcap_{i=1}^{k} \mathrm{cl}(T_{N_i}) = \mathrm{cl}\left(\bigcap_{i=1}^{k} T_{N_i}\right) = \mathrm{cl}\left(T_{\max_{i=1}^{k} N_i}\right) \neq \emptyset$$

for any finite $\{N_1, \ldots, N_k\} \subseteq \mathbb{N}$. So $\{\mathrm{cl}(T_N) \mid N \in \mathbb{N}\}$ has the finite intersection property. Therefore,

$$L := \bigcap_{N\in\mathbb{N}} \mathrm{cl}(T_N) \neq \emptyset\,.$$

Exercise: Show that any $x \in L$ is the limit of a subsequence of $(x_n)_{n\in\mathbb{N}}$!
The converse direction is more sophisticated. In order to keep this appendix short (we will see the generalized version in the lecture regarding *net-compactness*) we omit it. □